Instability of Hairy Black Holes in shift-symmetric Horndeski theories

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Introduction&Motivation

Hairy BH in shift-symmetric scalar-tensor theory

Introduction

BH hair in scalar tensor (ST) theory

No-hair theorem holds in many ST theories

BH hair
mass, charge, angular momentum

Brans-Dicke theory

$$\mathcal{L} = \frac{R}{2} - \frac{1}{2}(\partial\phi)^2 - U(\phi)$$

(in Einstein frame)

Hawking (1972); Bekenstein (1996)……

Covariant Galileon

$$\mathcal{L} \supset (\partial\phi)^2\Box\phi, \cdots$$

(spherically symmetric BHs)

Hui, Nicolis (2013)

and more…

However…

One consider shift-symmetric ST theory **with time-dependent scalar field**

→ BH solutions are found with non-trivial **scalar hair**

Bavichev, Charmousis(2014)

Bavichev, Charmousis(2014)

Dressing BH in shift-symmetric ST theory

Shift & reflection symmetry: $\phi \to \phi + \text{const.}, \quad \phi \to -\phi$

$$\mathcal{L} = [\zeta R - \eta(\partial\phi)^2 + \beta G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 2\Lambda]$$

$\zeta > 0, \eta, \beta : \text{const}$

$\Lambda :$ cosmological constant

Shift symmetry $\qquad$ EOM for scalar

$$\phi \to \phi + \text{const.} \longrightarrow \nabla_\mu J^\mu = 0$$

$$J^\mu = (\eta g^{\mu\nu} - \beta G^{\mu\nu})\partial_\nu\phi$$

Assumptions in Bavichev and Charmousis paper

$ds^2 = -A(r)dt^2 + \frac{1}{B(r)}dr^2 + r^2 d\Omega^2$ static and spherical symmetric

$J^r = 0 \longrightarrow$ Current $J^2 = J_\mu J^\mu$ regular at the horizon

$\phi(t,r) = qt + \psi(r) \longrightarrow$ **Space-time is static in shift-symmetric theory**

Dressing BH in shift-symmetric ST theory

Shift & reflection symmetry: $\phi \to \phi + \mathrm{const.}, \quad \phi \to -\phi$

$$\mathcal{L} = [\zeta R - \eta(\partial\phi)^2 + \beta G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 2\Lambda]$$

$\zeta > 0, \eta, \beta : \mathrm{const}$

Λ : cosmological constant

does not contain bare ϕ

contains derivative term $\partial_\mu\phi$

→ Time dependence term dose not appear in the theory.

(* We are not afraid that value of scalar field is unbound.)

$ds^2 = -A(r)dt^2 + \frac{1}{B(r)}dr^2 + r^2 d\Omega^2$ static and spherical symmetric

$J^r = 0$ → Current $J^2 = J_\mu J^\mu$ regular at the horizon

$\phi(t, r) = qt + \psi(r)$ → **Space-time is static in shift-symmetric theory**

Bavichev, Charmousis(2014)

Dressing BH in shift-symmetric ST theory

$$\mathcal{L} = [\zeta R - \eta(\partial\phi)^2 + \beta G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 2\Lambda]$$

$$\phi(t,r) = qt + \psi(r)$$

$$ds^2 = -A(r)dt^2 + \frac{1}{B(r)}dr^2 + r^2 d\Omega^2$$

Stealth Schwarzschild

$$A(r) = B(r) = 1 - \frac{\mu}{r} \qquad \mu : \text{const.}$$

$$\phi_\pm = qt \pm q\mu\left[2\sqrt{\frac{r}{\mu}} + \log\frac{\sqrt{r}-\sqrt{\mu}}{\sqrt{r}+\sqrt{\mu}}\right] + \phi_0$$

Self-tuned Schwarzschild-de-sitter

$$A(r) = B(r) = 1 - \frac{\mu}{r} + \boxed{\frac{\eta}{3\beta}}r^2 \longrightarrow \Lambda_{\text{eff}} = \boxed{-\frac{\zeta\eta}{\beta}} \neq \Lambda$$

This metric represent Schwarzschild BH in the presence of cosmological constant.

We do not conceive huge bare Λ through the metric.

Hairy BH solutions in the generalized theory

$$\mathcal{L} = [\zeta R - \eta(\partial\phi)^2 + \beta G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 2\Lambda] \qquad \phi(t,r) = qt + \psi(r)$$

Stealth Schwarzschild

$$A(r) = B(r) = 1 - \frac{\mu}{r} \qquad \mu : \text{const.}$$

$$\phi_\pm = qt \pm q\mu \left[2\sqrt{\frac{r}{\mu}} + \log\frac{\sqrt{r} - \sqrt{\mu}}{\sqrt{r} + \sqrt{\mu}} \right] + \phi_0$$

Self-tuned Schwarzschild-de-sitter

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Babichev, Charmousis(2014) can be generalized

$$\mathcal{L} = G_2(X) + G_4(X)R + G_{4X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right]$$

$$G_{4X} := \frac{\partial G_4}{\partial X} \qquad X := -\frac{1}{2}(\partial\phi)^2$$

Kobayashi, Tanahashi(2014)

The most general 2nd-order theory with shift & reflection symmetries

Hairy BH solutions in the generalized theory

$$\mathcal{L} = [\zeta R - \eta(\partial\phi)^2 + \beta G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 2\Lambda] \qquad \phi(t,r) = qt + \psi(r)$$

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$$\phi_\pm = qt \pm q\mu\left[2\sqrt{\frac{r}{\mu}} + \log\frac{\sqrt{r}-\sqrt{\mu}}{\sqrt{r}+\sqrt{\mu}}\right] + \phi_0$$

Many of found BHs are

X *constant solutions*

Self-tuned Schwarzschild-de-sitter

$$A(r) = B(r) = 1 - \frac{\mu}{r} + \frac{\eta}{3\beta}r^2 \rightarrow \Lambda_{\rm eff} = -\frac{\zeta\eta}{\beta} \neq \Lambda$$

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Kobayashi, Tanahashi(2014)

The most general 2nd-order theory with shift & reflection symmetries

Motivation

Stealth Schwarzschild sol and
Self-tuned Schwarzschild-de-sitter sol
are very interesting solutions.

How about stability of BHs?

Instability of Hairy BH in shift symmetric Horndeski theories

HO, T. Kobayashi, T. Suyama

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BH perturbations with **time-dependent scalar**

$$\mathcal{L} = G_2(X) + G_4(X)R + G_{4X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right] \qquad ds^2 = -A(r)dt^2 + \frac{1}{B(r)}dr^2 + r^2 d\Omega^2$$

$$\phi(t,r) = qt + \psi(r)$$

Basic Procedure

action 2nd-order in perturbations

$\downarrow$ **Hamiltonian analysis**

stability conditions

Set up

The most general 2nd-order theory with shift & reflection symmetries

$$\mathcal{L} = G_2(X) + G_4(X)R + G_{4X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right]$$

$$\phi(t,r) = qt + \psi(r)\ , \quad G_{4X} := \frac{\partial G_4}{\partial X}$$

Perturbations can be written as following eqs (odd-parity)

$$ds^2 = -A(r)dt^2 + \frac{1}{B(r)}dr^2 + r^2 d\Omega^2$$

$$E_{ab} = \sqrt{\det\gamma}\epsilon_{ab}$$

γ_{ab} two-dim metric on the sphere

ϵ_{ab} Levi-Civita symbol

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}$$

$$h_{tt} = 0, \quad h_{tr} = 0, \quad h_{rr} = 0$$

$$h_{ta} = \sum_{l,m} h_{0,lm}(t,r)E_{ab}\partial^b Y_{lm}(\theta,\varphi)$$

$$h_{ra} = \sum_{l,m} h_{1,lm}(t,r)E_{ab}\partial^b Y_{lm}(\theta,\varphi)$$

$$h_{ab} = \sum_{l,m} \cancel{h_{2,lm}}(t,r)[E_a{}^c\nabla_c\nabla_b Y_{lm}(\theta,\varphi) + E_b{}^c\nabla_c\nabla_a Y_{lm}(\theta,\varphi)]$$

gauge fixed (Regge-Wheeler gauge)

BH perturbations with **time-dependent scalar**

$$\mathcal{L} = G_2(X) + G_4(X)R + G_{4X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right] \qquad ds^2 = -A(r)dt^2 + \frac{1}{B(r)}dr^2 + r^2 d\Omega^2$$

$$\phi(t,r) = qt + \psi(r),\ X := -\frac{1}{2}(\partial\phi)^2$$

action 2nd-order in perturbations

Quadratic Lagrangian

$$\frac{2l+1}{2\pi}\mathcal{L}^{(2)} = A_1 h_0^2 + A_2 h_1^2 + A_4 h_0 h_1 + A_3\left(\dot{h}_1^2 - 2h_0'\dot{h}_1 + h_0'^2 + \frac{4}{r}h_0\dot{h}_1\right) \qquad \dot{h}_i := \frac{\partial h_i}{\partial t},\quad h_i' := \frac{\partial h_i}{\partial r}$$

$$A_1 = -\frac{l(l+1)(r^2A^2BA'G_4 - 2q^2r^2ABA'G_{4X} + \cdots}{A^{5/2}B^{1/2}}$$

$$A_1, A_2, A_3, A_4 \supset A(r), B(r), G_2, G_4, \cdots$$

BH perturbations with **time-dependent scalar**

$$\mathcal{L} = G_2(X) + G_4(X)R + G_{4X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right], \quad ds^2 = -A(r)dt^2 + \frac{1}{B(r)}dr^2 + r^2 d\Omega^2$$

$$\phi(t,r) = qt + \psi(r), \; X := -\frac{1}{2}(\partial\phi)^2$$

If we solve the perturbations…

$$\frac{2l+1}{2\pi}\mathcal{L}^{(2)} = A_1 h_0^2 + A_2 h_1^2 + A_4 h_0 h_1 + A_3\left(\dot{h}_1^2 - 2h_0'\dot{h}_1 + h_0'^2 + \frac{4}{r}h_0\dot{h}_1\right)$$

variation with respect to h_0

$$[A_3(h_0' - \dot{h}_1)]' = A_1 h_0 + A_4 h_1 + \frac{4}{r}A_3\dot{h}_1$$

constraint equation, non dynamical h_0

we cannot solve straightforwardly the constraint

BH perturbations with **time-dependent scalar**

$$\mathcal{L} = G_2(X) + G_4(X)R + G_{4X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right] \qquad ds^2 = -A(r)dt^2 + \frac{1}{B(r)}dr^2 + r^2 d\Omega^2$$

$$\phi(t,r) = qt + \psi(r),\ X := -\frac{1}{2}(\partial\phi)^2$$

field redefinition

$$\frac{2l+1}{2\pi}\mathcal{L}^{(2)} = A_1 h_0^2 + A_2 h_1^2 + A_4 h_0 h_1 + A_3\left(\dot{h}_1^2 - 2h_0'\dot{h}_1 + h_0'^2 + \frac{4}{r}h_0\dot{h}_1\right)$$

To remove non-dynamical h_0 we introduce a new field χ

$$\frac{2l+1}{2\pi}\mathcal{L}^{(2)} = \left(A_1 - \frac{2(rA_3)'}{r^2}\right)h_0^2 + A_2 h_1^2 + A_3\left[-\chi^2 + 2\chi\left(\dot{h}_1 - h_0' + \frac{2}{r}h_0\right)\right] + A_4 h_0 h_1$$

BH perturbations with time-dependent scalar

$$\mathcal{L} = G_2(X) + G_4(X)R + G_{4X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right] \qquad ds^2 = -A(r)dt^2 + \frac{1}{B(r)}dr^2 + r^2 d\Omega^2$$

$$\phi(t,r) = qt + \psi(r),\ X := -\frac{1}{2}(\partial\phi)^2$$

field redefinition

$$\frac{2l+1}{2\pi}\mathcal{L}^{(2)} = \left(A_1 - \frac{2(rA_3)'}{r^2}\right)h_0^2 + A_2 h_1^2 + A_3\left[-\chi^2 + 2\chi\left(\dot{h}_1 - h_0' + \frac{2}{r}h_0\right)\right] + A_4 h_0 h_1$$

$$h_0 = -\frac{2r\left\{2a_2\left[r(\chi a_3)' + 2\chi a_3\right] + r\dot{\chi}a_3 a_4\right\}}{4a_2[r^2 a_1 - 2\left(ra_3\right)'] - r^2 {a_4}^2},$$

$$h_1 = \frac{4a_3\dot{\chi}[r^2 a_1 - 2(ra_3)'] + 2ra_4[r(\chi a_3)' + 2a_3\chi]}{4a_2[r^2 a_1 - 2(ra_3)'] - r^2 {a_4}^2}.$$

$$\boxed{\frac{2l+1}{2\pi}\mathcal{L}^{(2)} = \frac{l(l+1)}{(l-1)(l+2)}\sqrt{\frac{B}{A}}\left(b_1\dot{\chi}^2 - b_2\chi'^2 + b_3\dot{\chi}\chi' - l(l+1)b_4\chi^2 - V(r)\chi^2\right)}$$

BH perturbations with **time-dependent scalar**

$$\mathcal{L} = G_2(X) + G_4(X)R + G_{4X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right], \quad ds^2 = -A(r)dt^2 + \frac{1}{B(r)}dr^2 + r^2 d\Omega^2$$

$$\phi(t,r) = qt + \psi(r), \; X := -\frac{1}{2}(\partial\phi)^2$$

Hamiltonian analysis

$$\frac{2l+1}{2\pi}\mathcal{L}^{(2)} = \frac{l(l+1)}{(l-1)(l+2)}\sqrt{\frac{B}{A}}\left(b_1\dot{\chi}^2 - b_2\chi'^2 + b_3\dot{\chi}\chi' - l(l+1)b_4\chi^2 - V(r)\chi^2\right)$$

$$\pi = \frac{\partial\tilde{\mathcal{L}}}{\partial\dot{\chi}}$$

$$H = \frac{1}{2}\int dr\sqrt{\frac{B}{A}}\left\{\frac{1}{b_1}\left(\sqrt{\frac{A}{B}}\pi - \frac{1}{2}b_3\chi'\right)^2 + b_2\chi'^2 + \left[l(l+1)b_4 + V\right]\chi^2\right\}$$

Hamiltonian must be positive define…

$$b_1 > 0, \quad b_2 > 0, \quad b_4 > 0$$

BH perturbations with **time-dependent scalar**

$$\mathcal{L} = G_2(X) + G_4(X)R + G_{4X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right], \quad ds^2 = -A(r)dt^2 + \frac{1}{B(r)}dr^2 + r^2 d\Omega^2$$

$$\phi(t,r) = qt + \psi(r), \; X := -\frac{1}{2}(\partial\phi)^2$$

Stability conditions

$$b_1 > 0, \quad b_2 > 0, \quad b_4 > 0$$

$$\mathcal{F} = 2\left[G_4 - \frac{q^2}{A}G_{4X}\right] > 0,$$

$$\mathcal{G} = 2\left[G_4 - 2XG_{4X} + \frac{q^2}{A}G_{4X}\right] > 0,$$

$$\mathcal{H} = 2\left(G_4 - 2XG_{4X}\right) > 0$$

Application to sample solution

ST theory: $\mathcal{L} = G_2(X) + G_4(X)R + G_{4X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right]$

$$\phi(t,r) = qt + \psi(r)\,,\ X := -\frac{1}{2}(\partial\phi)^2\,,\ G_{4X} := \frac{\partial G_4}{\partial X}\,,\ ds^2 = -A(r)dt^2 + \frac{1}{B(r)}dr^2 + r^2 d\Omega^2$$

Stealth sol, self-tuned de-sitter sol: $X = \mathrm{const.}$

$$\mathcal{F} = 2\left[\underbrace{G_4}_{\text{const}} - \frac{q^2}{A}G_{4X}\right] > 0,$$

$$\mathcal{G} = 2\left[\underbrace{G_4 - 2XG_{4X}}_{\text{const}} + \frac{q^2}{A}G_{4X}\right] > 0,$$

$$\mathcal{H} = 2\underbrace{(G_4 - 2XG_{4X})}_{\text{const}} > 0$$

Application to sample solution

ST theory: $\mathcal{L} = G_2(X) + G_4(X)R + G_{4X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right]$

$$\phi(t,r) = qt + \psi(r)\,,\; X := -\frac{1}{2}(\partial\phi)^2\,,\; G_{4X} := \frac{\partial G_4}{\partial X}\,,\; ds^2 = -A(r)dt^2 + \frac{1}{B(r)}dr^2 + r^2 d\Omega^2$$

Stealth sol, self-tuned de-sitter sol: $\boxed{X = \text{const.}}$

$$\mathcal{F} = 2\left[\underbrace{G_4}_{\text{const}} - \frac{q^2}{A}G_{4X}\right] > 0,$$

these terms are of opposite sign

$$\mathcal{G} = 2\left[\underbrace{G_4 - 2XG_{4X}}_{\text{const}} + \frac{q^2}{A}G_{4X}\right] > 0,$$

$$\mathcal{H} = 2\underbrace{\left(G_4 - 2XG_{4X}\right)}_{\text{const}} > 0$$

$\longrightarrow$ near the horizon $\quad \mathcal{F}\mathcal{G} \simeq -4\left(\frac{q^2}{A}G_{4X}\right)^2 < 0$

X=const solutions are unstable

Summary

Hairy BH solutions in shift-symmetric ST theory

Very interesting solutions are found

BH stability conditions

We obtain stability conditions (Hamiltonian analysis)

$$\mathcal{F} > 0, \mathcal{G} > 0, \mathcal{H} > 0$$

Hairy BH are unstable **due to time-dependent scalar**

$$X := -\frac{1}{2}(\partial\phi)^2$$ *=const. BH solutions are unstable*

Back up slides

No-hair theorem

BH hair in scalar tensor (ST) theory

Brans-Dicke theory

$$\mathcal{L} = \frac{R}{2} - \frac{1}{2}(\partial\phi)^2 - U(\phi)$$

Hawking (1972); Bekenstein (1996)……

static, Einstein flame

standard scalar-tensor theory hold no-hair theorem

Conformally coupled scalar field

$$\mathcal{L} = \frac{R}{2} - \frac{1}{2}(\partial\phi)^2 - \frac{1}{12}R\phi^2$$

BBMB solution   Bocharova, et.al (1970)

static, spherical symmetric,

scalar is **unbound**

Bavichev, Charmousis(2014)

Hairy BH solutions

$$\mathcal{L} = [\zeta R - \eta(\partial\phi)^2 + \beta G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 2\Lambda] \qquad \phi(t,r) = qt + \psi(r)$$

Stealth Schwarzschild

$$A(r) = B(r) = 1 - \frac{\mu}{r} \qquad \mu : \text{const.}$$

$$\phi_\pm = qt \pm q\mu\left[2\sqrt{\frac{r}{\mu}} + \log\frac{\sqrt{r}-\sqrt{\mu}}{\sqrt{r}+\sqrt{\mu}}\right] + \phi_0$$

Metric is Schwarzschild metric, but scalar field is non-trivial and regular at the horizon.

Consider Eddington-Finkelstein coordinates

$$v = t + \int (fh)^{-1/2} dr$$

$$\phi_+ = q\left[v - r + 2\sqrt{\mu r} - 2\mu\log\left(\sqrt{\frac{r}{\mu}} + 1\right)\right] + \text{const}$$

No-hair theorem

Covariant Galileon

$$\mathcal{L} \supset (\partial\phi)^2 \Box\phi, \cdots$$

Hui, Nicolis (2013)

static, spherically symmetric, asymptotically flat

There are two types of loopholes

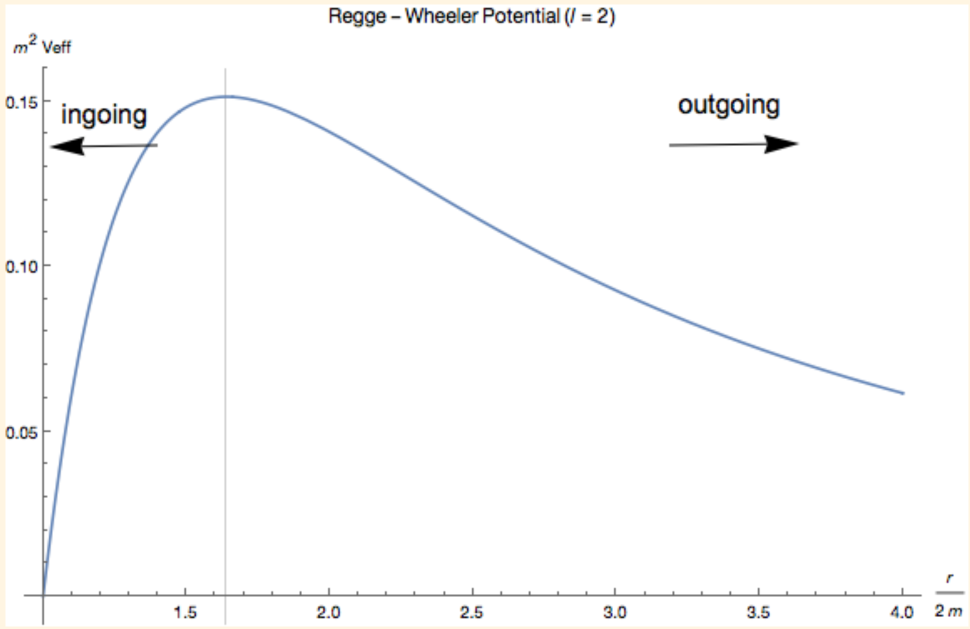
Gauss-Bonnet combination

Sotiriou, Zhou (2014)

lineally time dependence

Bavichev, Charmousis(2014)

Regge-Wheeler-Zerilli eq



Regge-Wheeler-Zerilli eqs

$$\frac{\partial^2 F(t,r)}{\partial t^2} - \frac{\partial^2 F(t,r)}{\partial r_*^2} + V(r)F(t,r) = 0$$

F satisfied the boundary right conditions

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} dr_* \left[\left|\frac{\partial F}{\partial t}\right|^2 + V\,|F|^2 + \left|\frac{\partial F}{\partial r}\right|^2\right] = -2(|\dot{f}|^2 + |\dot{g}|^2) < 0$$

F is decaying function or growing function

BH is stable or unstable

We focus on short length mode perturbations

Our stability conditions is necessary condition…

but classically and quantum mechanically unstable

BH perturbations with time-dependent scalar

$$\phi(t,r) = qt + \psi(r)\,, \quad ds^2 = -A(r)dt^2 + \frac{1}{B(r)}dr^2 + r^2 d\Omega^2$$

Hamiltonian

$$\frac{2l+1}{2\pi}\mathcal{L}^{(2)} = \frac{l(l+1)}{(l-1)(l+2)}\sqrt{\frac{B}{A}}(b_1\dot{\chi}^2 - b_2\chi'^2 + b_3\dot{\chi}\chi' - l(l+1)b_4\chi^2 - V(r)\chi^2)$$

we suppress l-factor

$$\widetilde{\mathcal{L}} = \frac{1}{2}\sqrt{\frac{B}{A}}\left\{b_1\dot{\chi}^2 - b_2\chi'^2 + b_3\dot{\chi}\chi' - [l(l+1)b_4 + V]\chi^2\right\}.$$

$$H = \frac{1}{2}\int dr\sqrt{\frac{B}{A}}\left\{\frac{1}{b_1}\left(\sqrt{\frac{A}{B}}\pi - \frac{1}{2}b_3\chi'\right)^2 + b_2\chi'^2 + [l(l+1)b_4 + V]\chi^2\right\}$$

To avoid

kinetic and radial instability $b_1 > 0, \quad b_2 > 0.$

large l instability $b_4 > 0$

$l=1$ mode: we rethink about gauge conditions

$$\frac{2l+1}{2\pi}\mathcal{L}^{(2)} = \frac{l(l+1)}{(l-1)(l+2)}\sqrt{\frac{B}{A}}(b_1\dot{\chi}^2 - b_2\chi'^2 + b_3\dot{\chi}\chi' - l(l+1)b_4\chi^2 - V(r)\chi^2)$$

$$l=1$$

$$h_{tt}=0, \quad h_{tr}=0, \quad h_{rr}=0$$

$$h_{ta} = \sum_{l,m} h_{0,lm}(t,r)E_{ab}\partial^b Y_{lm}(\theta,\varphi)$$

$$h_{ra} = \sum_{l,m} h_{1,lm}(t,r)E_{ab}\partial^b Y_{lm}(\theta,\varphi)$$

$$h_{ab} = \sum_{l,m} \cancel{h_{2,lm}}(t,r)[E_a{}^c\nabla_c\nabla_b Y_{lm}(\theta,\varphi) + E_b{}^c\nabla_c\nabla_a Y_{lm}(\theta,\varphi)]$$

automatically drop

Coefficients

$$\mathcal{L} = G_2(X) + G_4(X)R + G_{4X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right] \qquad ds^2 = -A(r)dt^2 + \frac{1}{B(r)}dr^2 + r^2 d\Omega^2$$

$$X := -\frac{1}{2}(\partial\phi)^2$$

$$\frac{2l+1}{2\pi}\mathcal{L}^{(2)} = A_1 h_0^2 + A_2 h_1^2 + A_4 h_0 h_1 + A_3\left(\dot{h}_1^2 - 2h_0'\dot{h}_1 + h_0'^2 + \frac{4}{r}h_0\dot{h}_1\right)$$

$$A_1 = \frac{l(l+1)}{r^2}\left[\frac{d}{dr}\left(r\sqrt{\frac{B}{A}}\mathcal{H}\right) + \frac{(l-1)(l+2)}{2\sqrt{AB}}\mathcal{F}\right],$$

$$A_2 = -\frac{(l-1)l(l+1)(l+2)}{2}\frac{\sqrt{AB}}{r^2}\mathcal{G},$$

$$A_3 = \frac{l(l+1)}{2}\sqrt{\frac{B}{A}}\mathcal{H},$$

$$A_4 = \frac{(l-1)l(l+1)(l+2)}{r^2}\sqrt{\frac{B}{A}}\mathcal{J}$$

$$\mathcal{F} = 2\left(G_4 - \frac{q^2}{A}G_{4X}\right),$$

$$\mathcal{G} = 2\left(G_4 - 2XG_{4X} + \frac{q^2}{A}G_{4X}\right),$$

$$\mathcal{H} = 2\left(G_4 - 2XG_{4X}\right),$$

$$\mathcal{J} = 2qG_{4X}\psi'$$