Causality and hyperbolicity of Lovelock theories of gravity

Harvey Reall

DAMTP, Cambridge University

HSR, N. Tanahashi & B. Way:
1406.3379 (CQG), 1409.3874 (PRD)
G. Papallo & HSR: 1508.05303 (JHEP)

Lovelock's theorem (1971)

Einstein's equation:

$$G_{ab} + \Lambda g_{ab} = 8\pi T_{ab}$$

LHS is most general symmetric tensor that is

- a function of g, ∂g, $\partial^2 g$
- divergence-free

This assumes $d = 4$ dimensions. For $d > 4$, extra terms can appear on LHS. These were determined by Lovelock.

Lovelock theories

$$E_{ab} = 8\pi T_{ab}$$

$$E^a{}_b \equiv \sum_{p \geq 0} k_p \delta^{ac_1 \ldots c_{2p}}_{bd_1 \ldots d_{2p}} R_{c_1 c_2}{}^{d_1 d_2} \ldots R_{c_{2p-1} c_{2p}}{}^{d_{2p-1} d_{2p}}$$

Antisymmetry: $p \leq [(d-1)/2]$

$$k_0 = \Lambda \qquad k_1 = -\frac{1}{4}$$

Einstein equation is obtained if we demand linearity in $\partial^2 g$, i.e., *quasilinearity*. General Lovelock theories are not quasilinear.

Why should I care about Lovelock theories?

- Wave equation $\Box\phi = 0$: can be (i) generalised to d dimensions; (ii) made nonlinear $\Box\phi = \mathcal{F}(\phi, \partial\phi, \partial^2\phi)$. There has been considerable interest in understanding such equations. Doing the same for Einstein equation gives Lovelock theories *uniquely*.
- Interesting mathematical question: how do properties of such theories differ from GR? Is GR special? Are Lovelock theories pathological in some way?

Causality in Lovelock theories

Causality of a PDE is determined by its *characteristic surfaces*.

In GR, a hypersurface is characteristic if, and only if, it is null so causality is determined by the lightcone.

Characteristic hypersurfaces of Lovelock theories are generically non-null (Aragone 1987, Choquet-Bruhat 1988) so gravity can propagate faster or slower than light

How do characteristic hypersurfaces behave in Lovelock theories?

Are Lovelock theories hyperbolic? (Necessary for well-posed initial value problem.)

We investigated this by looking at some particular solutions.

Ricci flat type N

Type N: $\exists$ null ℓ^a such that $\ell^a C_{abcd} = 0$ (e.g. pp-wave).
Solves Lovelock eq. of motion with $\Lambda = 0$.
A hypersurface is characteristic iff it is null w.r.t. one of $d(d-3)/2$ "effective" Lorentzian metrics $G_{(I)ab}$.

- Null cones of $G_{(I)ab}$ form a nested set, tangent along ℓ^a, causality determined by outermost cone
- Lovelock theories are hyperbolic in such backgrounds *for arbitrarily large curvature*, i.e., have the "right number" of characteristic surfaces

Static black holes

Lovelock theories admit static, spherically symmetric, solutions (Boulware & Deser 1985, Wheeler 1986)

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2$$

Characteristic hypersurfaces determined by two-derivative terms in eqs for linear perturbations (Dotti & Gleiser 2004-5, Takahashi & Soda 2009)

Linear perturbations can be classified into scalar, vector or tensor types; for each type there is an "effective metric". A surface is characteristic iff it is null w.r.t. one of these effective metrics. (NB: effective metrics non-generic!)

Static black holes

Large black hole (small curvature):

- Effective metrics Lorentzian, null cones form nested set outside event horizon $\Rightarrow$ equation of motion hyperbolic.
- Null cone of an effective metric can lie outside null come of g (gravity travels faster than light)

Small black hole: an effective metric can change signature near event horizon $\Rightarrow$ violation of hyperbolicity

Lovelock theories are not always hyperbolic: depends on background.

Dynamical hyperbolicity violation

Can one set up initial data so that theory is initially hyperbolic but becomes non-hyperbolic after some time?

Yes: consider *large* black hole: hyperbolicity violated to future of spacelike surface Σ *inside* black hole.

Is this generic? Generic linear perturbations cannot be evolved to future of Σ. Suggests nonlinear instability may ensure preservation of hyperbolicity (cf strong cosmic censorship). (G. Papallo)

Shock formation

"Speed of gravity" in Lovelock theories can vary in spacetime.

Can we make a wavepacket so that back of wavepacket travels faster than front?

cf compressible perfect fluid: speed of sound depends on pressure $\Rightarrow$ wave steepening $\Rightarrow$ shock!

We argued that this does indeed happen in Lovelock theories, but probably not at small curvature.

Resulting shocks would be naked but it may be possible to develop a theory of shock evolution, as for perfect fluid.

Future work

Local well-posedness of Lovelock theories for small curvature.

Nonlinear stability of Minkowski spacetime.

Generalization for $d = 4$ Horndeski theories.