Algebraically special solutions in five dimensions

Harvey Reall

DAMTP, Cambridge University

G. Bernardi de Freitas, M. Godazgar & HSR
CMP **340**, 291 (2015), CQG **33**, 095002 (2016)

Multiple WANDs

In 4d, a null vector field ℓ is a repeated principal null direction if, and only if,

$$\ell^b \ell_{[e} C_{a]b[cd} \ell_{f]} = 0$$

where C_{abcd} is the Weyl tensor.

For $d \geq 4$ dimensions, a null vector field ℓ satisfying this equation is called a *multiple Weyl aligned null direction* (multiple WAND). A spacetime admitting a multiple WAND is *algebraically special*. (Coley *et al* 2004, Ortaggio 2009).

Vacuum Einstein equation

$$R_{ab} = \Lambda g_{ab}$$

Existence of a multiple WAND makes this easier to solve. Can we determine all such solutions?

The optical matrix

Without loss of generality, ℓ can be assumed to be geodesic (Durkee & HSR 2009).

Let $\{\ell, n, m_i\}$ be a null basis, i.e.,

$$\ell \cdot n = 1 \qquad \ell \cdot m_i = n \cdot m_i = 0 \qquad m_i \cdot m_j = \delta_{ij}$$

The *optical matrix* of ℓ is

$$\rho_{ij} = \nabla_j \ell_i \equiv m_i^a m_j^b \nabla_b \ell_a$$

This can be decomposed into a trace, trace-free symmetric part, and antisymmetric part. These are the expansion, shear and rotation of the null geodesic congruence with tangent ℓ.

The canonical form for ρ_{ij}

$d = 4$: Goldberg-Sachs tells us that ρ_{ij} is shear-free, which implies that it has rank 2 or 0 (Kundt solutions).

$d = 5$: ρ_{ij} can have rank 3, 2, 1 or 0. Can choose m_i to bring ρ_{ij} to canonical form (analogue of $d = 4$ shear-free condition)
(Ortaggio, Pravda, Pravdova & HSR 2012)

$$b \begin{pmatrix} 1 & a & 0 \\ -a & 1 & 0 \\ 0 & 0 & 1+a^2 \end{pmatrix} \quad b \begin{pmatrix} 1 & a & 0 \\ -a & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad b \begin{pmatrix} 1 & a & 0 \\ -a & -a^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Rank 0: Kundt solutions.

Rank 1 optical matrix

L. Wylleman (2015): such solutions must be type D. Ortaggio *et al* (2012) showed that type D solutions must admit a *non-geodesic* multiple WAND. All solutions with this property were determined by Durkee & HSR (2009):

- Products $dS_3 \times S^2$ and $AdS_3 \times H^2$
- Analytically continued Schwarzschild

$$ds^2 = f d\tau^2 + f^{-1} dr^2 + r^2 ds_3^2$$

where ds_3^2 is 3d maximally symmetric Lorentzian spacetime. Includes "Kaluza-Klein bubble" and "AdS soliton".

Rank 3 optical matrix

If $\rho_{[ij]} = 0$ then $\rho_{ij} \propto \delta_{ij}$, i.e, Robinson-Trautman. Such solutions were classified by Podolsky & Ortaggio (2006): just Schwarzschild solution.

$\rho_{[ij]} \neq 0$: direct integration of Einstein equation, two families of solutions (Bernardi de Freitas, Godazgar & HSR 2015)

- 3-parameter local solution, 3d abelian symmetry group. For certain parameter ranges this is the Myers-Perry black hole with unequal angular momenta.
- 2-parameter local solution, cohomogeneity-1. For certain parameter ranges this is the Myers-Perry black hole with equal angular momenta.

This is a novel uniqueness theorem for the Myers-Perry solution. All solutions are type D.

Rank 2 optical matrix

The optical matrix has kernel m_4. One can introduce local coordinates $(u, r, x, z, \bar{z})$ with $\ell = \partial/\partial r$, $m_4 = \partial/\partial x$.

Integration of the Einstein equation determines explicit dependence of metric on r and x. Einstein equation reduces to a system of PDEs in $(u, z, \bar{z})$, just as for 4d algebraically special solutions. Solutions divide into 3 classes.

Class 1: warped products

$$ds^2 = dx^2 + F(x)^2 ds_4^2$$

where ds_4^2 is a 4d non-Kundt algebraically special vacuum solution. Any type D solution belongs to this family (Wylleman 2016?).

Rank 2 continued

Class 2:

$$\begin{aligned} ds^2 &= \left(dx + Ydz + \bar{Y}d\bar{z} - J\ell\right)^2 + 2\ell\left(dr + Udz + \bar{U}d\bar{z} - \mathcal{H}\ell\right) \\ &+ \frac{2(r^2+\Sigma^2)}{P^2}dzd\bar{z} \qquad\qquad \ell = -(du + Ldz + \bar{L}d\bar{z}) \end{aligned}$$

Includes ($\Lambda = 0$) Kaluza-Klein uplifts of 4d algebraically special Einstein-Maxwell solutions.

Class 3:

$$\begin{aligned} ds^2 &= (dx + Ydz + \bar{Y}d\bar{z} - J\ell)^2 \\ &+ F(x)^2\left[2\ell\left(dr + Udz + \bar{U}d\bar{z} - \mathcal{H}\ell\right) + \frac{2(r^2+\Sigma^2)}{P^2}dzd\bar{z}\right] \end{aligned}$$

ℓ as in class 2.

Open questions

Classification of 5d algebraically special solutions "solved" in the same sense as for 4d (reduced to PDEs in 3 coordinates).

Does the rank 2 case contain physically interesting examples that are not warped products or Kaluza-Klein spacetimes?

Can we now tackle $d > 5$?