Black hole non-modal linear stability: the $\Lambda \geq 0$ Schwarzschild case

talk based on PRL **112** (2014) 191101 and arXiv 1603.03749

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July 13th, 2016

Dedicated to Sergio Dain

Defining linear stability

Linear vacuum Einstein equation with cosmological constant (LEE)

$$\mathcal{E}[\delta g_{\alpha\beta}] := -\frac{1}{2}\nabla^{\gamma}\nabla_{\gamma}\delta g_{\alpha\beta} - \frac{1}{2}\nabla_{\alpha}\nabla_{\beta}(g^{\gamma\delta}\delta g_{\gamma\delta}) + \nabla^{\gamma}\nabla_{(\alpha}\delta g_{\beta)\gamma} - \Lambda\, \delta g_{\alpha\beta} = 0$$

We are interested in equivalence classes $[\delta g_{\alpha\beta}]$ of slns mod gauge transformations:

$$\delta g_{\alpha\beta} \sim \delta g_{\alpha\beta} + \nabla_{\alpha}\xi_{\beta} + \nabla_{\beta}\xi_{\alpha}$$

Rough stability idea: The outer region of a stationary black hole is linearly stable if linear metric perturbations *do not grow unbounded.*
Expectation: *perturbed Kerr BH metrics decay to a nearby Kerr BHs.*

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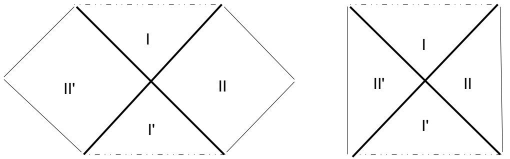
- If $\Lambda \geq 0$ both *II* and $II \cup II' \cup I \cup I'$ are globally hyperbolic.
- If $\Lambda < 0$ boundary conditions have to be specified at the conformal timelike boundary and there are instabilities if some Robin boundary conditions are chosen.

Carter-Penrose diagram: II outer static region; I: inner non static region



Zero or positive cosmological constant   Negative cosmological constant

Modal approach: i) solving the LEE

$$\underbrace{g_{\alpha\beta}dz^\alpha dz^\beta}_{\nabla_\alpha,\ \epsilon_{\alpha\beta\gamma\delta}} = \underbrace{g_{ab}(x)dx^a dx^b}_{\text{orbit space: } D_c,\ \epsilon_{ab}} + r^2(x)\underbrace{\hat{g}_{ij}(y)dy^i dy^j}_{S^2:\ \hat{D}_k,\ \epsilon_{ij}}$$

$$= \overbrace{-f(r)dt^2 + \frac{dr^2}{f(r)}} + r^2\,\overbrace{\left(d\theta^2 + \sin^2(\theta)\, d\phi^2\right)}$$

$$\mathbf{J}^2 = \sum_{k=1}^{3} (\pounds_{J_k})^2, \quad (J_3 = \partial_\phi, \text{ etc }) \quad \text{and} \quad P : (t, r, \theta, \phi) \to (t, r, \pi - \theta, \phi + \pi)$$

$$\delta g_{\alpha\beta} = \sum_{p=\pm,\ell,m} \delta g_{\alpha\beta}^{(\ell,m,p)} \text{ where } \mathbf{J}^2 \delta g_{\alpha\beta}^{(\ell,m,\pm)} = -\ell(\ell+1)\delta g_{\alpha\beta}^{(\ell,m,\pm)}$$

even (+) and odd (-) modes $P_* \delta g_{\alpha\beta}^{(\ell,m,\pm)} = \pm(-1)^\ell \delta g_{\alpha\beta}^{(\ell,m,\pm)}$

$\ell \geq 2$ LEE equivalent to infinite set of 1+1 wave eqns: $\underbrace{g^{ab} D_a D_b \phi^\pm_{(\ell,m)} - U^\pm_\ell \phi^\pm_{(\ell,m)} = 0}_{\text{LEE for } \ell \geq 2\text{modes}}$

Modal approach: i) solving the LEE

	odd modes ($p=-$)	even modes ($p=+$)
$\ell=0$		δM shift within Kerr family
$\ell=1$	$\delta j^{(k)}$ shift within Kerr family	
$\ell\geq 2$	$\delta g^{(\ell,m,-)}_{\alpha\beta}=\mathcal{D}^{(-)}_{\alpha\beta}[\phi^{-}_{(\ell,m)},S_{(\ell,m)}]$	$\delta g^{(\ell,m,+)}_{\alpha\beta}=\mathcal{D}^{(+)}_{\alpha\beta}[\phi^{+}_{(\ell,m)},S_{(\ell,m)}]$

$S_{(\ell,m)}$ are spherical harmonics: $\hat{g}^{ij}\hat{D}_i\hat{D}_jS_{(\ell,m)}=-\ell(\ell+1)S_{(\ell,m)}$,

$\phi^{\pm}_{(\ell,m)}$ are solutions of 2D wave eqns: $\underbrace{g^{ab}D_aD_b\phi^{\pm}_{(\ell,m)}-U^{\pm}_{\ell}\phi^{\pm}_{(\ell,m)}=0}_{\text{LEE for }\ell\geq 2\text{modes}}$

Generic perturbations parametrized by gauge invariant fields and constants: $\delta M,\delta j^{(k)}$ (constants read from initial datum) and $\phi^{\pm}_{(\ell,m)}$ (dynamical fields)

Odd sector non modal approach: linear stability

- Standard approach to stability problem consists on setting bounds on the infinite set of fields $\phi_{(\ell,m)}(t,r)$
- Up to 2 derivatives of these fields enter $\delta g_{\alpha\beta}$, thus 4 derivatives in $\delta R^{\alpha}{}_{\beta\gamma\delta}$.
- Any geometric implies $\sum_{(\ell,m)}$
- Implications of the boundedness (either integral or pointwise) of the $\phi^{\pm}_{(\ell,m)}$ on the perturbed geometry not obvious a priori
- Look to parametrize the space $\mathcal{L}$ of linearized solutions of the Einstein's equations around Schwarzschild de Sitter BH with geometrical fields as an alternative to

$$\mathcal{L} = \{\delta M, \delta j^{(k)}, \phi^{\pm}_{(\ell,m)}\}$$

- Need to estimate the growth of these fields in order to analyze stability

RW equation and 4D-RW equation:

$$\ell \geq 2 \text{ LEE } \Leftrightarrow \overbrace{\left[\partial_t^2 - \partial_{r*}^2 + f\left(\frac{\ell(\ell+1)}{r^2} - \frac{6M}{r^3}\right)\right]\phi^{-}_{(\ell,m)} = 0}^{\text{RW equation}}, \; \hat{D}^k \hat{D}_k S_{(\ell,m)} = -\ell(\ell+1) S_{(\ell,m)}$$

$$\Leftrightarrow \underbrace{\left[\nabla_\alpha \nabla^\alpha + \frac{8M}{r^3} - \frac{2\Lambda}{3}\right]\Phi = 0}_{\text{4D RW equation]}} \quad \Phi = \sum_{(\ell\geq 2,m)} \frac{\phi^{-}_{(\ell,m)}}{r} S_{(\ell,m)}$$

Odd sector non modal approach: perturbed curvature scalars

We consider the effect of a perturbation on the curvature scalars:

$$Q_- = \frac{1}{48}{}^*C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta}, \qquad Q_+ = \frac{1}{48}C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta}, \qquad X = \frac{1}{720}\left(\nabla_\epsilon C_{\alpha\beta\gamma\delta}\right)\left(\nabla^\epsilon C^{\alpha\beta\gamma\delta}\right).$$

The background values are:

$$Q_{-\,S(A)dS} = 0, \quad Q_{+\,S(A)dS} = \frac{M^2}{r^6}, \quad X_{S(A)dS} = \frac{M^2}{r^9}(r-2M) - \frac{\Lambda M^2}{3r^6}$$

This implies that the following fields are gauge invariant:

$$G_- = \delta Q_- \quad \text{and} \quad G_+ = (9M - 4r + \Lambda r^3)\delta Q_+ + 3r^3\delta X$$

$G_- = G_-[\delta g^{(-)}]$, this functional depends on up to four derivatives of the $\phi^-_{(\ell,m)}$.

Using repeatedly the LEE we arrive at: (recall that $\mathcal{L} = \{\overbrace{\delta j^{(k)}, \phi^-_{(\ell,m)}}^{odd}, \overbrace{\delta M, \phi^+_{(\ell,m)}}^{even}\}$)

$$G_- = -\frac{6M}{r^7}\sqrt{\frac{4\pi}{3}}\sum_{m=1}^{3}\delta j^{(m)} S_{(1,m)} - \underbrace{\frac{3M}{r^5}\sum_{(\ell\geq 2,m)}\frac{(\ell+2)!}{(\ell-2)!}\frac{\phi^-_{(\ell,m)}}{r}S_{(\ell,m)}}_{\mathbf{J}^2(\mathbf{J}^2+2)\Phi}$$

Odd sector non modal approach: linear stability

$$G_- = -\frac{6M}{r^7}\sqrt{\frac{4\pi}{3}}\sum_{m=1}^{3}\delta j^{(m)} S_{(1,m)} - \frac{3M}{r^5}\sum_{(\ell\geq 2,m)} \frac{(\ell+2)!}{(\ell-2)!}\frac{\phi^-_{(\ell,m)}}{r} S_{(\ell,m)}$$

From G_- we can recover $(\delta j^{(m)}, \phi^-_{(\ell,m)})$ and therefore $\delta g^{(-)}_{\alpha\beta}$ in a given gauge

All the gauge invariant information in $\delta g^{(-)}_{\alpha\beta}$ is encoded in G_-

$[\delta g^{(-)}_{\alpha\beta}] \to G_- \left([\delta g^{(-)}_{\alpha\beta}]\right)$ is a bijection

Odd LEE are entirely equivalent to $\left[\nabla_\alpha \nabla^\alpha + \frac{8M}{r^3} - \frac{2\Lambda}{3}\right](r^5 G_-) = 0$!!! (4DRWE)

Boundedness: For any smooth solution of the odd LEE which has compact support on Cauchy surfaces of the extended $\mathrm{I} \cup \mathrm{II} \cup \mathrm{I}' \cup \mathrm{II}'$ Schwarzschild (Schwarzschild de Sitter) BH, there exists a constant K_- such that $|G_-| < K_-\ r^{-6}$ for $r > r_h$ ($r_h < r < r_c$)

Decay: For large t Price/Brady tails give a slowly rotating Kerr/dS BH:

$$G_- \simeq -\frac{6M}{r^7}\sqrt{\frac{4\pi}{3}}\sum_{m=1}^{3}\delta j^{(m)} S_{(1,m)}$$

Even perturbations: difficulties for a 4D approach

1) For odd perturbations

$$\ell \geq 2 \text{ odd LEE } \Leftrightarrow [\partial_t^2 \overbrace{-\partial_{r*}^2 + f\left(\frac{\ell(\ell+1)}{r^2} - \frac{6M}{r^3}\right)}^{\mathcal{H}_\ell^- \text{ (RW Hamiltonian)}}]\phi_{(\ell,m)}^- = 0, \ \hat{D}^k \hat{D}_k S_{(\ell,m)} = -\ell(\ell+1) S_{(\ell,m)}$$

$$\Leftrightarrow \underbrace{\left[\nabla_\alpha \nabla^\alpha + \frac{8M}{r^3} - \frac{2\Lambda}{3}\right] \Phi = 0}_{\text{4D RW equation]}} \quad \Phi = \sum_{(\ell \geq 2, m)} \frac{\phi_{(\ell,m)}^-}{r} S_{(\ell,m)}$$

2) For even perturbations the 2D Zerilli wave equation is not related to a 4D equation:

$$\mathcal{H}_\ell^+ = -\partial_{r*}^2 + f \ V_{(\ell,m)}^Z \quad \text{(Zerilli Hamiltonian)}$$

with potential ($\mu = (\ell - 1)(\ell + 2)$)

$$V_{(\ell,m)}^Z = \frac{[\mu^2 \ell(\ell+1) - 24M^2\Lambda] r^3 + 6\mu^2 M r^2 + 36\mu M^2 r + 72 M^3}{r^3 \ (6M + ((\ell-1)(\ell+2))^2 r)^2}.$$

Even perturbations: difficulties for a 4D approach

3) For odd perturbations there is a gauge invariant scalar related to the perturbed Weyl tensor (thus expected to satisfy some kind of wave equation)

$$G_- = \delta Q_- = \delta \left(\tfrac{1}{48} {}^*C^{\alpha\beta\gamma\delta} C_{\alpha\beta\gamma\delta} \right)$$

4) For even perturbations we need to use differential invariants to construct curvature related gauge invariant perturbation fields. The simplest such field is

$$G_+ = G_+ = (9M - 4r + \Lambda r^3)\delta Q_+ + 3r^3 \delta X$$

$$Q_+ = \tfrac{1}{48} C^{\alpha\beta\gamma\delta} C_{\alpha\beta\gamma\delta}, \qquad X = \tfrac{1}{720} \left(\nabla_\epsilon C_{\alpha\beta\gamma\delta}\right) \left(\nabla^\epsilon C^{\alpha\beta\gamma\delta}\right).$$

These are not expected to satisfy wave equations as a consequence of the LEE !

Even perturbations: Chandrasekhar's even $\leftrightarrow$ odd duality

Chandrasekhar (80's) found that

$$\mathcal{H}_\ell^{\pm} = \mathcal{D}_\ell^{\pm}\mathcal{D}_\ell^{\mp} - E_\ell{}^2, \;\; \mathcal{D}_\ell^{\pm} = \pm\frac{\partial}{\partial r^*} + W_\ell(r)$$

$\mathcal{D}_\ell^{+}$ maps solutions of the odd (RW) equations to solutions of the even (Z) equation:

$$(\partial_t^2 + \mathcal{H}_\ell^{-})\phi^{-}_{(\ell,m)} = 0 \Rightarrow (\partial_t^2 + \mathcal{H}_\ell^{+})(\mathcal{D}_\ell^{+}\phi^{-}_{(\ell,m)}) = 0$$

- The general solution of the differential equation $\mathcal{D}_\ell^{+}\chi = 0$ is a constant times

$$\chi_\ell^{-} = \frac{(\ell+2)(\ell-1)r + 6M}{r}\exp(-w_\ell\, r^*) \quad \notin L^2(\mathbb{R}, dr^*)$$

- This implies that $\mathcal{D}_\ell^{+} : L^2(\mathbb{R}, dr^*) \to L^2(\mathbb{R}, dr^*)$ is injective (also in more general spaces, since the above sln blows up at the BS).

Assume $\Lambda \geq 0$. For any solution $\phi^{+}_{(\ell,m)}$ of the ZE in $L^2(\mathbb{R}_{r^*}, dr^*)$ there is a unique solution $\phi^{-}_{(\ell,m)}$ of RWE in $L^2(\mathbb{R}_{r^*}, dr^*)$ such that $\phi^{+}_{(\ell,m)} = \mathcal{D}_\ell^{+}\phi^{-}_{(\ell,m)}$.

Even perturbations: Non modal stability

$$G_+ = -\frac{2M\delta M}{r^5} + \frac{M}{2r^4}\sum_{\ell\geq 2}\frac{(\ell+2)!}{(\ell-2)!}\left[f\partial_r + Z_\ell\right]\phi^+_{(\ell,m)}S_{(\ell,m)}, \quad Z_\ell = Z_\ell(r)$$

From G_+ we can recover $(\delta M, \phi^+_{(\ell,m)})$ and therefore $\delta g^{(+)}_{\alpha\beta}$ in a given gauge

All the gauge invariant information in $\delta g^{(+)}_{\alpha\beta}$ is encoded in G_+

$[\delta g^{(+)}_{\alpha\beta}] \to G_+\left([\delta g^{(+)}_{\alpha\beta}]\right)$ is a bijection

$$G_+ = -\frac{2M\dot{M}}{r^5} + \frac{M}{2r^3}\left[\partial_t^2\Phi_5 + \sum_{j=0}^{4} r^{-j}\Phi_j + \frac{f(r-3M)}{r^3}\Phi_5 + \frac{f}{r}\Phi_6\right] + \frac{M}{2r^3}\partial_{r^*}\Phi_6 + \frac{M(r-3M)}{2r^5}\partial_{r^*}\Phi_5,$$

(using Chandra's duality) $\Phi_j = \sum_{(\ell\geq 2,m)} P_j(\ell)\frac{\phi^-_{(\ell,m)}}{r}S_{(\ell,m)}, \quad j = 0,\ldots 6$ satisfy the 4DRWE !!!

Boundedness: For any smooth solution of the odd LEE which has compact support on Cauchy surfaces of the extended $\mathrm{I}\cup\mathrm{II}\cup\mathrm{I}'\cup\mathrm{II}'$ Schwarzschild (or Schwarzschild de Sitter) BH, there exists a constant K_+ such that $|G_+| < K_+\, r^{-3}$ for $r > r_h$ ($r_h < r < r_c$)

Summary

- All gauge invariant perturbation info contained in scalar curvature perturbation fields

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- $[\delta g_{\alpha\beta}] \to (G_-([\delta g_{\alpha\beta}]), G_+([\delta g_{\alpha\beta}]))$ is a bijection
- LEE entirely equivalent to 4D RW equation !

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$$|G_-| \leq K_-/r^6, \quad |G_+| \leq K_+/r^3$$

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- $[\delta g_{\alpha\beta}] \to (G_-([\delta g_{\alpha\beta}]), G_+([\delta g_{\alpha\beta}]))$ is a bijection
- LEE entirely equivalent to 4D RW equation !
- Perturbations are bounded in the outer region:

$$|G_-| \leq K_-/r^6, \quad |G_+| \leq K_+/r^3$$

- For large t we get decay within Kerr/ds family:

$$G_- \simeq -\tfrac{6M}{r^7}\sqrt{\tfrac{4\pi}{3}}\sum_{m=1}^{3}\delta j^{(m)} S_{(1,m)}, \qquad G_+ \simeq -\frac{2M\delta M}{r^5}$$