Recent Progress in the Black Hole Stability Problem

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July 11th, 2016 *GRG21, Columbia University, New York*

Overview

1. Introduction to the Black Hole Stability Problem
2. Poor Man's Linear Stability of Kerr: $\Box_{g_{M,a}}\psi = 0$
3. The Linear Stability of the Schwarzschild metric
4. The case of $\Lambda \neq 0$: Stability of Kerr-dS and Kerr-AdS

The Kerr family of solutions $g_{M,a}$

This is a two-parameter family of solutions of

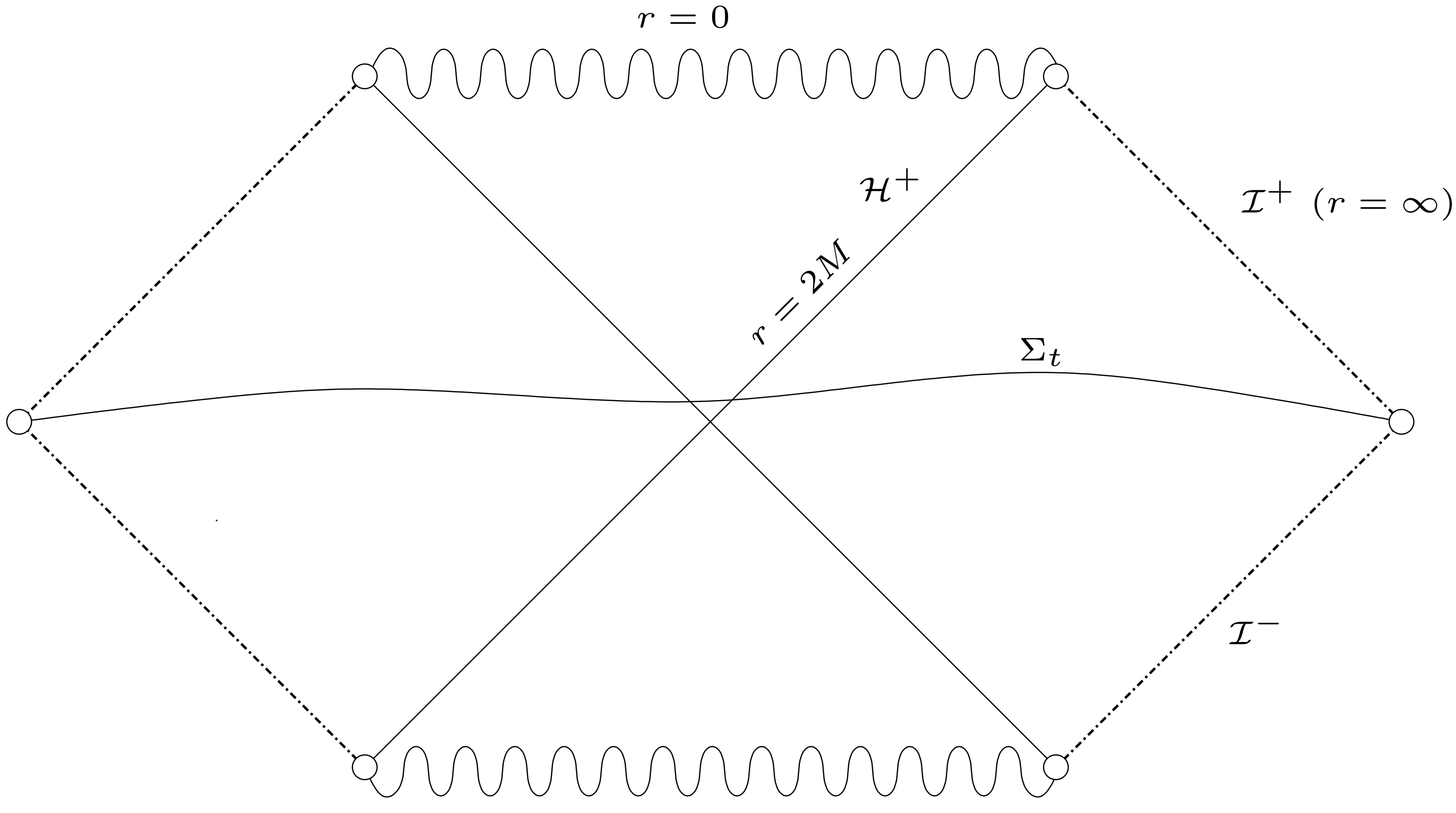
$$Ric\,[g_{M,a}] = 0$$

written down explicitly in 1963.

- stationary, axisymmetric, asymptotically flat black holes
- for $a = 0$ it reduces to the famous Schwarzschild solution
- generalisations to $Ric\,[g_{M,a,\Lambda}] = \Lambda \cdot g_{M,a,\Lambda}$:
  $\Lambda > 0$ (Kerr-de Sitter) and $\Lambda < 0$ (Kerr-anti de Sitter)

Schwarzschild spacetime (1916)

$$g_M = -\left(1-\frac{2M}{r}\right)dt^2+\left(1-\frac{2M}{r}\right)^{-1}dr^2+r^2\left(d\theta^2+\sin^2\theta d\phi^2\right)$$



Two famous problems regarding the Kerr metric:

- **Uniqueness** of the Kerr family among stationary solutions
$\rightarrow$ Carter-Robinson (1973), Hawking (1973), Chruściel-Costa (2008), Alexakis–Ionescu–Klainerman (2009); *proven near Kerr*
- **Dynamical Stability** of the Kerr family
completely open!

I will talk about the second.

Notions of Stability

1. Full **non-linear stability** (Cauchy problem)

   $$\left(\Sigma^{(3)}, h, K\right) + \text{constraints} \rightarrow \left(\mathcal{M}^{3+1}, g\right) \quad \text{satisfying} \quad Ric\,[g] = 0$$

   Stability here means stability of the exterior region!

2. **Linear stability**: Solutions to the linearised equations remain

   (a) uniformly bounded

   (b) decay

   on the black hole exterior.

3. **Mode stability**

   There exist no exponentially growing mode solutions.

What is known for the Kerr family

1. non-linear stability is open
2. is open in general. For the Schwarzschild subfamily we proved

   Theorem (Dafermos, GH, Rodnianski 2016)**.** *Linear stability in the sense of 2a) and 2b) holds for the Schwarzschild subfamily.*

   The Kerr case is work in progress.

3. mode stability is known
   Whiting (1989), Shlapentokh-Rothman (2014)
   $\rightarrow$ see talk of C. Paganini on Thursday

The nature of the problem from the PDE perspective

The Einstein vacuum equations

$$R_{\mu\nu}\left[g\right] = 0 \tag{1}$$

for a Lorentzian metric g can be viewed as a complicated coupled system of non-linear wave equations.

Indeed, in so-called harmonic coordinates (1) becomes

$$\Box_g g_{\mu\nu} = Q\left(\partial g, \partial g\right) \tag{2}$$

<u>GOAL:</u> **Understand dynamics of (1) near the Kerr family.**

The **key mechanism** for stability is **decay**.

One typically proceeds along the following lines

1. Prove linear stability, i.e. 2a) and 2b)
   with both robust and sufficiently strong decay estimates

2. Understand the structure of the non-linearity (null condition)

N.B.: Near flat space: "Stability of Minkowski space"
(Christodoulou–Klainerman '90)

Let us focus on **Linear Stability**.

1. The poor man's version: $\Box_{g_{M,a}}\psi = 0$.
   $\rightarrow$ crucial to understand decay mechanisms
2. The linearised Einstein equations
   (a) approaches using **decoupling**
   (Regge-Wheeler, Zerilli, Teukolsky)
   (b) approaches using the **canonical energy**
   (Hollands–Wald 2012 and collaborators; $\rightarrow$ talk of S. Green)

Note that to attack 2. one needs to face the issue of gauge.

Poor man's Linear Stability

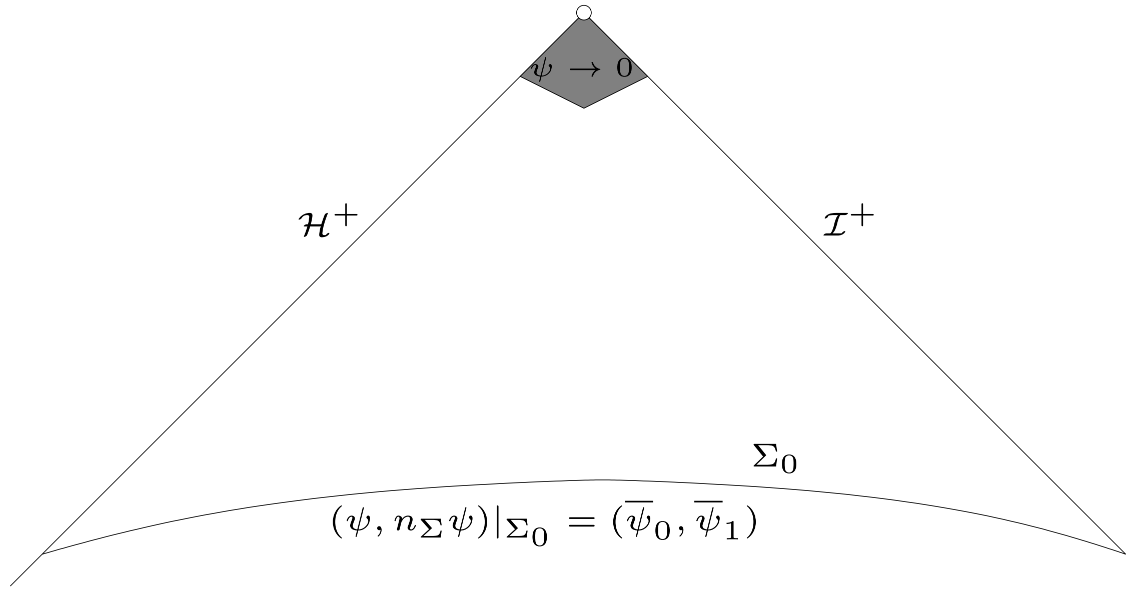
Regarding the poor man's version, we have

Theorem (Dafermos–Rodnianski–Shlapentokh-Rothman (2014))**.** *Solutions of the linear wave equation*

$$\Box_{g_{M,a}}\psi = 0 \tag{3}$$

for $g_{M,a}$ a subextremal member ($|a| < M$) of the Kerr-family decay inverse polynomially in time on the black hole exterior.

Previous work (2005-2014): Dafermos–Rodnianski, Blue–Sterbenz, Tataru-Tohaneanu, Andersson–Blue; Kay–Wald (1987)



The key geometric phenomena that need to be understood:

1. Redshift effect near $\mathcal{H}^+$
2. Trapped null-geodesics
3. Superradiance
4. the coupling of 1. – 3.

The extremal case: $a = M$

The previous theorem **fails** if the black hole background is the **extremal** Kerr solution due to the *Aretakis instability* (2012).

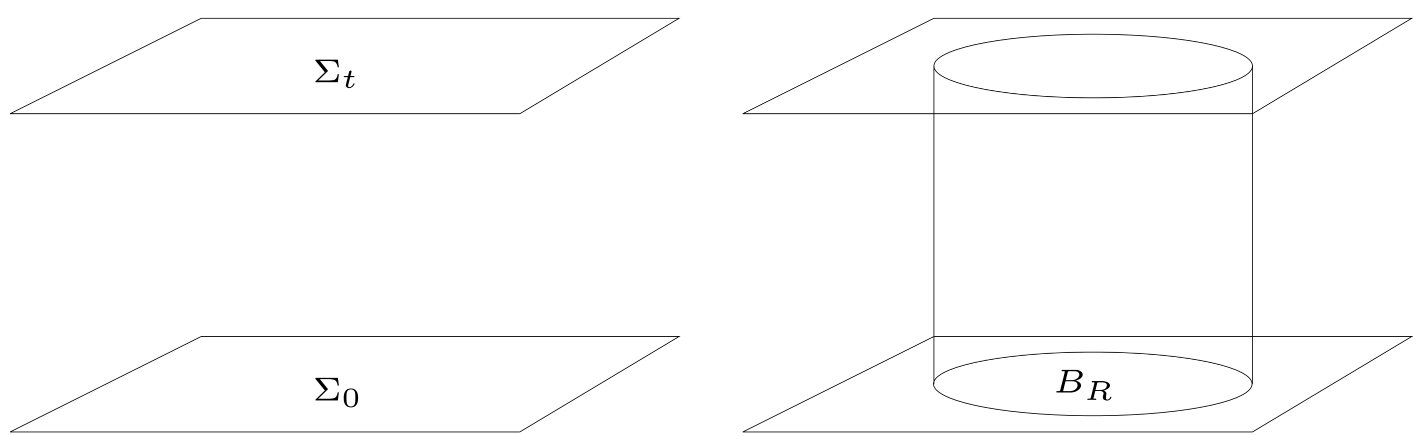
- hierarchy of conservation laws along the event horizon
- generalised to gravitational perturbations + numerics for non-linear models Lucietti–Murata–Reall–Tanahashi (2012-2013)
- wide range of applications:
  $\rightarrow$ forthcoming work of Aretakis, Angelopoulos and Gajic

Aside: Four Slides about the Estimates

Recall Minkowski space: $\Box_\eta \psi = 0$. Two key estimates

$$\int_{\Sigma_t} (\partial_t \psi)^2 + |\nabla \psi|^2 = \int_{\Sigma_0} (\partial_t \psi)^2 + |\nabla \psi|^2 \quad \text{energy conservation}$$

$$\int_0^T dt \int_{\Sigma_t \cap \{r \le R\}} (\partial_t \psi)^2 + |\nabla \psi|^2 \le C_R \int_{\Sigma_0} (\partial_t \psi)^2 + |\nabla \psi|^2 \quad \text{ILED}$$



Already in the Schwarzschild case deriving analogues of these two estimates requires

- understanding of the *redshift* near $\mathcal{H}^+$ to prove boundedness
- understanding of *trapping* at the photon sphere to prove decay

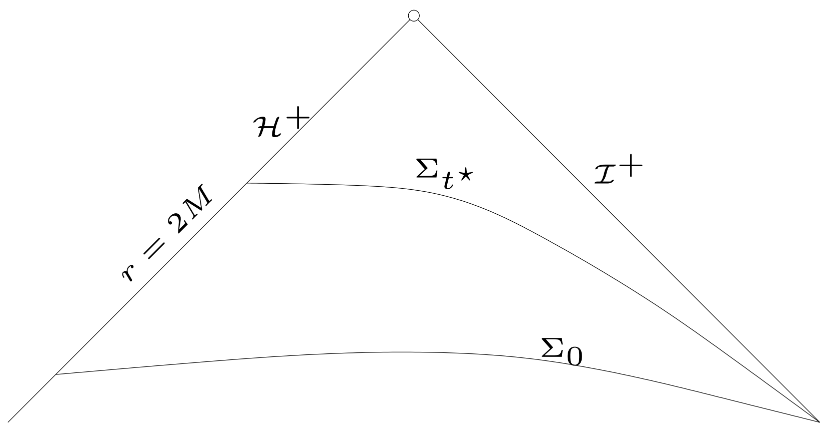
The Kerr case is much more complicated.

The redshift in Schwarzschild

The static Killing vector field $T = \partial_t$ gives rise to

$$\boxed{\int_{\Sigma_{t^\star}} |\partial\psi|^2 \left(1 - \frac{2M}{r}\right) \lesssim \int_{\Sigma_0} |\partial\psi|^2 \left(1 - \frac{2M}{r}\right)}$$

with $|\partial\psi|^2$ denoting (the sum of) all derivatives of ψ.



The redshift effect is about removing the degeneracy at $r = 2M$ to get a <u>non-degenerate</u> boundedness statement.

Trapping in Schwarzschild

- Null-geodesics can orbit in the timelike hypersurface $r = 3M$: This is the photon sphere.
- In the high frequency approximation, solutions to the wave equation travel along null-geodesics!

$\implies$ Non-degenerate decay estimates for Schwarzschild are necessarily associated with a loss of derivatives. (Sbierski 2013)

$$\boxed{\int_0^T dt^\star \int_{\Sigma_{t^\star}\cap\{r\leq R\}} |\partial\psi|^2 \leq C_R \int_{\Sigma_0} |\partial\psi|^2 + |\partial^2\psi|^2}$$

Linear Stability of Schwarzschild

The linearization procedure

Consider a one-parameter family of Lorentzian metrics in double null-coordinates

$$\boldsymbol{g}\left(\epsilon\right) \doteq -4\boldsymbol{\Omega}^2\left(\epsilon\right)dudv + \boldsymbol{\not g}_{CD}\left(\epsilon\right)\left(d\theta^C - \boldsymbol{b}^C\left(\epsilon\right)dv\right)\left(d\theta^D - \boldsymbol{b}^D\left(\epsilon\right)dv\right)$$

with $\boldsymbol{g}\left(0\right)$ being the Schwarzschild metric of mass M.

- $\boldsymbol{\Omega}\left(\epsilon\right) = \Omega + \epsilon\overset{(1)}{\Omega} + \mathcal{O}\left(\epsilon^2\right)$, etc.
- express curvature components and connection-coefficients in associated null-frame
- write down above equations to order ϵ

The result is a complicated system of equations of

- linearised null-curvature components (Newman-Penrose scalars) satisfying the Bianchi equations
- linearised connection coefficients satisfying transport equations

I won't show this system. Evolution is well-posed.

We would like to show boundedness and decay of *all* linearised quantities in terms of initial data of the linearised system.

The system of linearised Einstein equations: Key-observations

1. special solutions: pure gauge solutions
2. special solutions: linearised Kerr solutions
3. hierarchy of decoupled gauge invariant quantities

Gauge invariant quantities which decouple

It has long been known that the gauge invariant null-curvature components $\overset{(1)}{\alpha}$ and $\underline{\overset{(1)}{\alpha}}$ satisfy decoupled wave equations: The **Teukolsky** (or Bardeen-Press '73) **equation**:

$$\boxed{\Box_g \overset{(1)}{\alpha} + \left(1 - \frac{3M}{r}\right) \partial_t \overset{(1)}{\alpha} + V \overset{(1)}{\alpha} = 0.}$$

Only mode stability but not even uniform boundedness was known.

Hierarchy of gauge invariant quantities

There exists a second order differential operator which when applied to α or $\underline{\alpha}$ yields new quantities

$$P := \mathfrak{D}^2 \overset{\scriptscriptstyle(1)}{\alpha} \quad , \quad \underline{P} := \underline{\mathfrak{D}}^2 \underline{\overset{\scriptscriptstyle(1)}{\alpha}} \tag{4}$$

- the quantities P and $\underline{P}$ satisfy the Regge-Wheeler equation, which *does* admit both a good energy estimate and an ILED! All the poor man's theory for $\Box_g \phi = 0$ applies.
- the quantities P and $\underline{P}$ control α, $\underline{\alpha}$ respectively, in particular decay for P and $\underline{P}$ implies decay for α and $\underline{\alpha}$.

These transformations appear at the level of mode solutions in the work of Chandrasekhar.

Corollary. *(Dafermos–GH–Rodnianski 2016) Solutions to the Teukolsky equation decay inverse polynomially in time.*

Note this result holds independently of the whole system of gravitational perturbations.

Previous and related work

- Moncrief (1975), Martel-Poisson (2005), Sarbach–Tiglio (2001), Dotti (2014) (metric perturbations; Regge-Wheeler and Zerilli)
- Finster–Smoller (2009, 2016)

From gauge invariant to all geometric quantities

- can show $\boxed{\overset{(1)}{\alpha} = \underline{\overset{(1)}{\alpha}} = 0}$ globally $\Longrightarrow$ solution is pure gauge
- need *quantitative* estimates of all geometric quantities
- This can be done. Remarkably, one can only show boundedness but not decay for some of the quantities. Why is that?

Solution decays to a pure gauge solution which is <u>dynamically determined</u> and can be <u>quantitatively estimated from data</u>.

Theorem (DHR 2016; Linear Stability of Schwarzschild)**.**
General solutions $\mathscr{S}$ *of the system of gravitational perturbations on Schwarzschild arising from suitably normalised characteristic initial data*

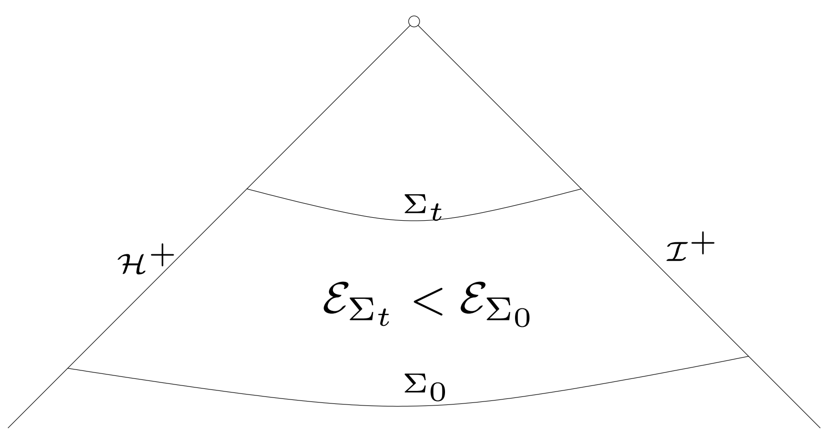
- *remain uniformly bounded on the black hole exterior and in fact*
- *decay inverse polynomially to a linearised Kerr solution* $\mathscr{K}$ *after adding to* $\mathscr{S}$ *a dynamically determined pure gauge solution* $\mathscr{G}$ *which is itself uniformly bounded by initial data.*

$\rightarrow$ non-linear applications

The canonical energy (à la Hollands–Wald; Friedman)

For g_0 static (or station.-axisym.) look at perturbation $g_0 + \epsilon \cdot \gamma$.
There exists a quantity $\mathcal{E}_\Sigma = \int_\Sigma \partial\gamma \cdot \partial\gamma$ which

1. satisfies a conservation law
2. $\mathcal{E}_\mathcal{H}$ and $\mathcal{E}_\mathcal{I}$ are manifestly positive (controlling linearised shears)
3. $\mathcal{E}_\Sigma$ is gauge invariant (modulo boundary terms)
4. $\mathcal{E}_\Sigma = 0$ iff γ is a perturbation to another stationary solution



If $\mathcal{E}_\Sigma$ could be shown to be non-negative, one would obtain a priori control on the energy flux through the horizon and null-infinity from initial data. Such estimates are potentially very useful.

Theorem (GH 2016)**.** *Such a priori control can indeed be obtained for the system of gravitational perturbations on Schwarzschild.*

Idea: Work entirely with double null-foliation and exploit the freedom to add pure gauge solutions.

Recent developments for $\Lambda > 0$ and $\Lambda < 0$

The case $\Lambda > 0$

Non-linear stability of Kerr-de Sitter was very recently proven by Hintz–Vasy (2016) for small a.

Previously the poor man's model $\Box_{g_{M,a,\Lambda}}\psi = 0$ (as well as tensorial and non-linear versions thereof) had been understood
Bony–Häfner (2007), Dafermos–Rodnianski (2007), Dyatlov 2010-2013, Hintz–Vasy 2013-2015

- *exponential decay* of the energy makes problem easier
- mode stability plays a crucial role
- more in Claude Warnick's talk on Wednesday

The case $\Lambda < 0$

Poor man's $\boxed{\Box_{g_{M,a,\Lambda}}\psi = 0}$ understood (reflecting bdy conditions)

1. Hawking–Reall bound satisfied
   $\rightarrow$ logarithmic decay (GH-Smulevici 2011-2013)

2. Hawking–Reall bound violated
   $\rightarrow$ exponentially growing modes (Dold 2015)

The superradiant instability in 2. has been studied successfully for gravitational perturbations via the canonical energy (Green, Ishibashi, Hollands, Wald 2014-2015)

Current work in the physics literature: Endstate of superradiant instability, Non-linear Instabilities arising from slow logarithmic decay, Instability of AdS ($\rightarrow$ talks of Dias, Way, Santos today)

That would be the topic of another talk!

Thank you for your attention.