Quantum Vacuum Polarization in Large Dimensional Spacetimes

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1 - Introduction

- Calculations of $\langle\phi^2\rangle$ are usually restricted to four dimensions, where the counterterms necessary to obtain a renormalized quantity are well known. Calculations in larger dimensions (D>4) are very few (Frolov & Mazzitelli, Shiraishi & Maki, Thompson & Lemos, Decanini & Folacci, Breen et al) due to the increased complexity of the counterterms, and are restricted to the horizon.

- The goal of this work is to extended the calculation of $\langle\phi^2\rangle$ to higher dimensional black holes, everywhere outside the horizon, which is important when one is interested in higher dimensional theories.

- The case D=5 will be analyzed in full detail, where standard QFT methods will be used to obtain a regularized result. The D=6 case will also be analyzed, in the large mass limit.

Green function of a scalar field in a D-dimensional spacetime

2 - Green function of a scalar field in a D-dimensional spacetime

The vacuum polarization is, by definition,

$$\langle \phi^2(x) \rangle = \lim_{x' \to x} i\, G_F(x, x') = \lim_{x' \to x} G_E(x, x')$$

which satisfies the Green function equation in a curved space background

$$\left(\Box - m^2 - \xi R\right) G_E(x, x') = -\frac{\delta^{(D)}(x - x')}{\sqrt{g}}$$

We will be particularly interested in the case of a Schwarzschild-Tangherlini spacetime

$$ds^2 = f(r) d\tau^2 + \frac{1}{f(r)} dr^2 + r^2 d\Omega^2_{D-2} \qquad f(r) = 1 - \left(\frac{2M}{r}\right)^{D-3}$$

Since the problem has spherical symmetry, we can start by expanding the solution as a combination of the hyperspherical harmonics (N=D-3)

$$G_E(x, x') = \frac{\alpha}{4\pi^2} \frac{\Gamma\left(\frac{N}{2}\right)}{\pi^{\frac{N}{2}}} \sum_{n=-\infty}^{\infty} e^{i\omega_n(\tau - \tau')} \sum_{l=0}^{\infty} \left(l + \frac{N}{2}\right) C_l^{\left(\frac{N}{2}\right)} (\Omega \cdot \Omega')\, G_{nl}(r, r') \qquad \omega_n = \alpha n \qquad \alpha = 2\pi T$$

2 - Green function of a scalar field in a D-dimensional spacetime

$$\frac{d^2G_{nl}}{dr^2} + \left(\frac{N+1}{r} + \frac{f'}{2f}\right)\frac{dG_{nl}}{dr} - \left(\frac{\omega_n^2}{f^2} + \frac{l(l+N)}{fr^2} + \frac{m^2+\xi R}{f}\right)G_{nl} = -\frac{\delta(r-r')}{fr^{N+1}}$$

Let $\chi^+_{nl}(r)$ and $\chi^-_{nl}(r)$ be two solutions of the homogenous equation which are regular at infinity and at the horizon, respectively. Then a solution to the non-homogeneous differential equation is

$$G_{nl}(r,r') = -\frac{1}{fr^{N+1}}\frac{\chi^+_{nl}(r_<)\,\chi^-_{nl}(r_>)}{\chi^+_{nl}(r')\frac{d\chi^-_{nl}}{dr}(r) - \frac{d\chi^+_{nl}}{dr}(r')\chi^-_{nl}(r)}$$

We use a WKB ansatz for the solutions of the homogeneous equation

$$\chi^{\pm}(r) = \frac{e^{\pm\int_{r_h}^{r} W(u)/f(u)\,du}}{\sqrt{r^{N+1}W(r)}}$$

which reduces the radial Green function to the form

$$G_{nl}(r,r) = \frac{1}{2\,r^{N+1}\,W(r)}$$

2 - Green function of a scalar field in a D-dimensional spacetime

The homogeneous differential equation becomes

$$W^2 = \Phi + a_1 \frac{W'}{W} + a_2 \frac{W'^2}{W^2} + a_3 \frac{W''}{W}$$

with

$$\Phi = \left[\left(l + \frac{N}{2} \right)^2 - \frac{N^2}{4} \right] \frac{f}{r^2} + \omega_n^2 + f\mu^2 \qquad a_1 = \frac{ff'}{2} \quad a_2 = -\frac{3}{4} f^2 \quad a_3 = \frac{r^2}{2} \frac{f}{g}$$

$$\mu^2 = m^2 + \xi R + \left(\frac{N}{2} + \frac{1}{2} \right) \frac{f'}{2} + \left(\frac{N^2}{4} - \frac{1}{4} \right) \frac{f}{r^2}$$

Inserting the WKB ansatz in the Green function and taking the coincidence limit, we obtain ($\alpha = 2\pi/\beta$)

$$G_E(x,x) = \frac{\alpha}{8\pi^2 r} \frac{1}{(\pi r^2)^{\frac{N}{2}}} \frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma(N)} \sum_{n=-\infty}^{\infty} \sum_{l=0}^{\infty} \frac{\Gamma(l+N)}{\Gamma(l+1)} \left(l + \frac{N}{2} \right) \frac{1}{W(r)}$$

2.1 – Regularization of the coincidence limit

We now express the solution iteratively as $W = W^{(0)} + W^{(1)} + \cdots$, with the solution written as

$$\frac{1}{W_{nl}} = \frac{1}{\Phi^{1/2}} \left(1 + \delta_1 \Phi + \delta_2 \Phi + \cdots \right)$$

At first order, for example, we obtain

$$\delta_1 \Phi = -\frac{a_1}{4} \frac{\Phi'}{\Phi^2} + \left(\frac{a_3 - a_2}{8} \right) \frac{\Phi'^2}{\Phi^3} - \frac{a_3}{4} \frac{\Phi''}{\Phi^2}$$

For large l, it is known that there will be non-physical divergences, where denoted $\mathcal{T}_l$ which can always be rearranged in the form

$$\mathcal{T}_l = \frac{\alpha}{8\pi^2 r} \frac{1}{(\pi r^2)^{\frac{N}{2}}} \frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma(N)} \sum_n e^{in\alpha\varepsilon} \sum_l R_l(r)$$

For each N, there will be a function $R_l(r)$ that will cancel the divergences in the angular modes. A regularized result in the l modes is thus

$$G_E(x,x) = \frac{\alpha}{8\pi^2 r} \frac{1}{(\pi r^2)^{\frac{N}{2}}} \frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma(N)} \sum_{n=-\infty}^{\infty} \sum_{l=0}^{\infty} \left\{ \frac{\Gamma(l+N)}{\Gamma(l+1)} \left(l + \frac{N}{2} \right) \frac{1}{W(r)} - \mathcal{T}_l \right\}$$

2.1 – Regularization of the coincidence limit

The expression is still divergent in the full coincidence limit $\varepsilon \to 0$ due to loop divergences. The physical divergent part of the Green function for a general D-dimensional spacetime can be obtained by using the Schwinger-DeWitt expansion, and is given by

$$G_{E\,\mathrm{div.}}(x,x') = \frac{2\Delta^{1/2}}{(4\pi)^{D/2}} \sum_{k=0}^{k_D} a_k(x,x')(2m^2)^{\nu}|z|^{-\nu}K_{\nu}(|z|)$$

$$z = -2m^2\sigma$$

As the dimensionality of the spacetime increases, the number of relevant heat kernel coefficients a_k increases as well. For D=4 and 5, only k=0 and 1 are necessary. For D=6 and 7, k=2 is needed as well, and so on.

- The complexity of the heat kernel coefficients rapidly increases as well (they have only been calculated up to k=4).

- The calculation of the counterterms directly from the Green function provides a way to compute these expressions using computer symbolic manipulation.

Vacuum polarization for D=5

3.1 – Summation over l and n modes

The renormalized vacuum polarization is given by the renormalized Green function

$$\langle \phi^2(x) \rangle = \lim_{x' \to x} \left[G_E(x,x') - G_{E\,\text{div.}}(x,x') \right]$$
$$= \lim_{x' \to x} \left[G_E^{WKB}(x,x') - G_{E\,\text{div.}}(x,x') + \delta G \right]$$

It is instructive to confirm explicitly the cancellation of the divergent terms. From

$$G_E(x,x) = \frac{\alpha}{4\pi^3 r^3} \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} \left\{ \frac{(l+1)^2}{W_{nl}} - \frac{(l+1)r}{\sqrt{f}} \right\} = \frac{\alpha}{4\pi^3 r^3} \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} J_n^{(5)}(l)$$

the divergent part can be calculated analytically by use of the Abel-Plana sum formula

$$G_E(x,x) = \frac{\alpha}{4\pi^3 r^3} \sum_{n=1}^{\infty} \Bigg(\underbrace{\frac{J_n^{(5)}(0)}{2}}_{\mathcal{P}_1} + \underbrace{\int_0^{\infty} J_n^{(5)}(x)dx}_{\mathcal{P}_2} + \underbrace{i \int_0^{\infty} \frac{J_n^{(5)}(ix) - J_n^{(5)}(-ix)}{e^{2\pi x} - 1} dx}_{\mathcal{P}_3} \Bigg)$$

where we only need to consider WKB up to first order, i.e.

$$\frac{1}{W_{nl}} = \frac{1 + \delta\Phi}{\sqrt{\Phi}}$$

3.1 – Summation over l and n modes

We can evaluate the divergent piece of each term

$$\mathrm{div}\left[\mathcal{P}_1\right] = \sum_{n=1}^{\infty} \frac{1}{2\omega_n}$$

$$\mathrm{div}\left[\mathcal{P}_2\right] = \sum_{n=1}^{\infty} \left\{ -\frac{1}{3\omega_n} + \frac{r^3}{8\sqrt{f}} \left[\frac{4\omega_n^2}{f} + \left(m^2 - [a_1] + \frac{f'}{r} - \frac{f'^2}{4f} + \frac{f''}{3} \right) \right] \left(1 + \ln\left(\frac{r\omega_n}{2\sqrt{f}} \right) \right) \right\}$$

$$\mathrm{div}\left[\mathcal{P}_3\right] = \sum_{n=1}^{\infty} \left(-\frac{1}{6\omega_n} \right)$$

which leads to the total divergent piece

$$\mathrm{div}\left[\mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3\right] = \sum_{n=1}^{\infty} \left\{ \frac{r^3}{8\sqrt{f}} \left[\frac{4\omega_n^2}{f} + \left(m^2 - [a_1] + \frac{f'}{r} - \frac{f'^2}{4f} + \frac{f''}{3} \right) \right] \left(1 + \ln\left(\frac{r\omega_n}{2\sqrt{f}} \right) \right) \right\}$$

This must be compared to the divergent terms obtained from the Schwinger-DeWitt expansion.

3.1 – Summation over l and n modes

Using the point splitting method and the Schwinger-DeWitt expansion, and choosing a point separation in the temporal direction, we obtain

$$\sigma = \frac{f}{2}\varepsilon^2 - \frac{f f'^2}{96}\varepsilon^4 + O(\varepsilon^6)$$

which leads to the counterterms

$$G_{E\mathrm{div.}} = \frac{1}{8\pi^2 f^{3/2}}\frac{1}{\varepsilon^3} + \frac{1}{16\pi^2\sqrt{f}}\left(m^2 - [a_1] + \frac{f'}{4r} - \frac{f'^2}{16 f} + \frac{f''}{12}\right)\frac{1}{\varepsilon}$$

To compare with the counterterms expressed as a sum in the energy modes, we use the results

$$\sum_{n=1}^{\infty} \log\left(a\omega_n\right)\cos\left(\omega_n \epsilon\right) = -\frac{\pi}{2\alpha}\frac{1}{\varepsilon} + O(\epsilon)$$

$$\sum_{n=1}^{\infty} \omega_n^2 \log\left(a\omega_n\right)\cos\left(\omega_n \epsilon\right) = \frac{\pi}{2\alpha}\frac{1}{\varepsilon^3} + O(\epsilon)$$

which exactly cancels all the divergent terms.

Figure 1: Profile of the renormalized vacuum polarization for $M_{BH} = 5$.

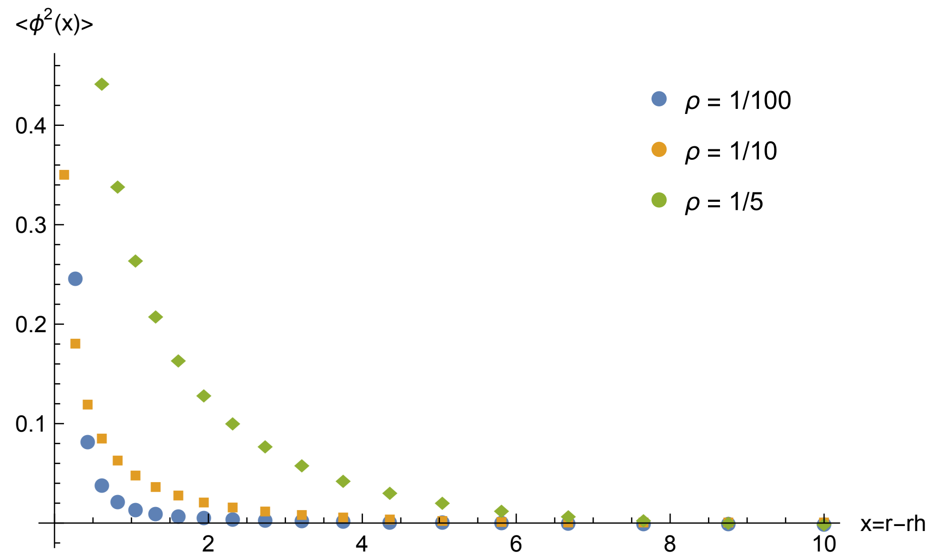
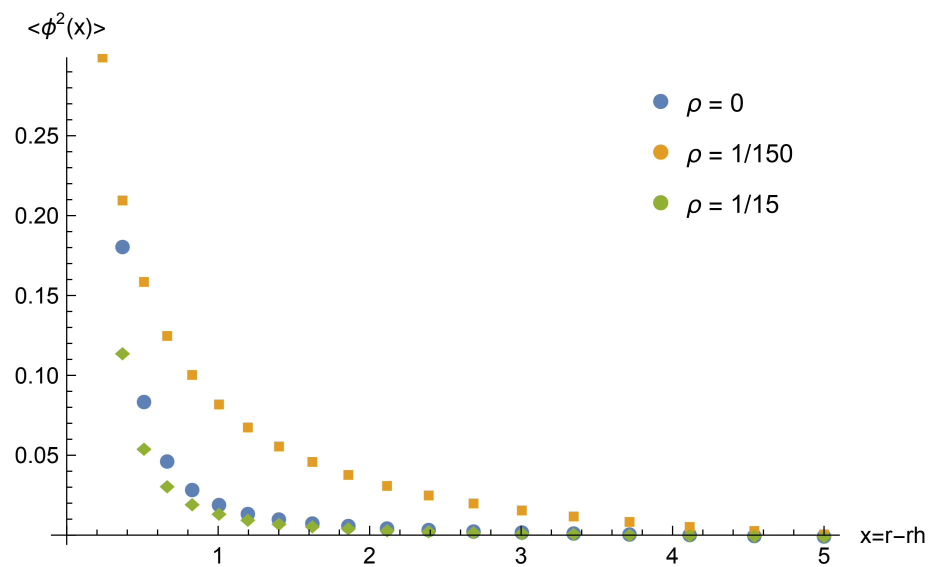


Figure 1: Profile of the renormalized vacuum polarization for $M_{BH} = 15/2$.



$$\rho = \frac{m}{2M_{BH}}$$

$$f(r) = 1 - \left(\frac{2M}{r}\right)^2$$

Vacuum polarization for D=6

4.1 – Summation over l and n modes

For D=6, the Green function is

$$G_E(x,x) = \frac{\alpha}{4\pi^4 r^4} \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} \left\{ \frac{(l+2)(l+1)(l+3/2)}{W_{nl}} - \mathcal{T}_l \right\} = \frac{\alpha}{4\pi^4 r^4} \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} J_n^{(6)}(l)$$

where we must now also consider the second order WKB approximation

$$\frac{1}{W_{nl}} = \frac{1 + \delta_1\Phi + \delta_2\Phi}{\sqrt{\Phi}}$$

$$\begin{aligned}
\delta_2\Phi = {} & \frac{11{a_1}^2{\Phi'}^2}{32\Phi^4} - \frac{{a_1}^2\Phi''}{8\Phi^3} + \frac{17a_1a_2{\Phi'}^3}{32\Phi^5} - \frac{a_1a_2\Phi'\Phi''}{4\Phi^4} - \frac{a_1a_3\Phi'''}{4\Phi^3} - \frac{25a_1a_3{\Phi'}^3}{32\Phi^5} \\
& + \frac{19a_1a_3\Phi'\Phi''}{16\Phi^4} + \frac{27{a_2}^2{\Phi'}^4}{128\Phi^6} - \frac{{a_2}^2{\Phi'}^2\Phi''}{8\Phi^5} - \frac{51a_2a_3{\Phi'}^4}{64\Phi^6} - \frac{a_2a_3{\Phi''}^2}{8\Phi^4} \\
& + \frac{41a_2a_3{\Phi'}^2\Phi''}{32\Phi^5} - \frac{a_2a_3\Phi'''\Phi'}{4\Phi^4} + \frac{75{a_3}^2{\Phi'}^4}{128\Phi^6} - \frac{{a_3}^2\Phi'''}{8\Phi^3} + \frac{15{a_3}^2{\Phi''}^2}{32\Phi^4} \\
& - \frac{45{a_3}^2{\Phi'}^2\Phi''}{32\Phi^5} + \frac{{a_3}^2\Phi'''\Phi'}{2\Phi^4}
\end{aligned}$$

4.1 – Summation over l and n modes

One can verify that, in the large mass limit, the divergent piece is

$$G_{E\text{div.}} = \sum_{n=1}^{\infty}\left(\left\{\frac{\alpha m^4}{64\pi^3} + \frac{\alpha m^2}{32\pi^3}\left(\frac{1}{6}-\xi\right)\left(f'' + \frac{8f'}{r} + \frac{12f}{r^2} - \frac{12}{r^2}\right)\right\}\frac{1}{\omega_n} + \frac{\alpha m^2}{16\pi^3}\frac{\omega_n}{f}\right)$$

On the other hand, using the Schwinger-DeWitt expansion, we obtain in the same limit

$$G_{E\text{div.}} = \left(\frac{m^4}{64\pi^3} - \frac{[a_1]m^2}{32\pi^3}\right)\log\varepsilon - \frac{m^2}{16\pi^3 f}\frac{1}{\varepsilon^2}$$

Using the relations

$$\log\varepsilon = -\alpha\sum_{n=1}^{\infty}\frac{\cos(\omega_n\varepsilon)}{\omega_n} + c_1 + O(\varepsilon) \qquad \frac{1}{\varepsilon^2} = -\alpha\sum_{n=1}^{\infty}\omega_n\cos(\omega\varepsilon) + c_2 + O(\varepsilon^2)$$

one is able to explicitly confirm the cancellation of the divergent terms.

5 – Conclusions

- Although widely studied in 4 dimensions, the vacuum polarization in higher dimensional spacetimes had not been tackled outside the horizon up until now.

- We fully studied the case of a 5-dimensional Schwarzschild-Tangherlini spacetime, and obtained a renormalized quantity. The regularity of the result was explicitly proven by direct calculation of the counterterms. The 6-dimensional case was also studied, where regularity was shown for the large mass limit.

- Although the WKB approach has proven itself very useful in a variety of situations, it becomes clear that for higher dimensional spacetimes the problem becomes more complicated, due to the necessity of including higher order corrections as the dimensionality increases.