A lower bound for the mass of multiple charged rotating black holes

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The Penrose heuristic argument

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For a Kerr-Newman black hole we have:

$$m_i^2 = \frac{A_i}{16\pi} + \frac{q_i^2}{2} + \frac{\pi(q_i^4 + 4\mathcal{J}_i^2)}{A_i}$$

or minimizing over A_i:

$$m_i^2 \geq \frac{q_i^2 + \sqrt{q_i^4 + 4\mathcal{J}_i^2}}{2},$$

with equality iff the Kerr-Newman is *extreme*.

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$$\sqrt{2}m = \sqrt{2}\sum m_i \geq \sum \sqrt{q_i^2 + \sqrt{q_i^4 + 4\mathcal{J}_i^2}} \geq \sqrt{q^2 + \sqrt{q^4 + 4\mathcal{J}^2}}.$$

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Lemma

Let $a_i, b_i \in \mathbb{R}$ and let $a = \sum a_i$, $b = \sum b_i$. Then

$$\left(a^4 + b^2\right)^{1/4} \leq \sum \left(a_i^4 + b_i^2\right)^{1/4}.$$

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m, q, J inequality for multiple black holes

Theorem (Khuri-W)

Let (M, g, k, E, B) be a smooth, simply connected, axially symmetric, maximal initial data set satisfying $\mu_{EM} \geq 0$ and $g(J_{EM}, \eta) = 0$, and with $N+1$ ends, one designated AF and the others either AF or AC. Then

$$m \geq \mathcal{F}(\mathcal{J}_1, \dots, \mathcal{J}_N, q_1^e, \dots, q_N^e, q_1^b, \dots, q_N^b). \qquad (1)$$

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Conjecture

Equality cannot be achieved if $N > 1$ unless all charges are of the same sign and the angular momenta vanish. In this case, the initial data set is isometric to the canonical slice of a MP spacetime.

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- J. Costa, *Proof of a Dain inequality with charge*, J. Phys. A, **43** (2010), no. 28, 285202.

Sketch of proof

Under the hypotheses, one may define EM potentials χ, ψ, and a twist potential v, so that

$$m \geq \mathcal{M}(U, v, \chi, \psi) = \frac{1}{8\pi}\int_{\mathbb{R}^3}\left(|\nabla U|^2 + \frac{e^{4U}}{\rho^4}|\nabla v + \chi\nabla\psi - \psi\nabla\chi|^2 \right. \\ \left. + \frac{e^{2U}}{\rho^2}\left(|\nabla\chi|^2 + |\nabla\psi|^2\right)\right)\, dx$$

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If we define $u = U - \log \rho$, then

$$\mathcal{H}_\Omega(u, v, \chi, \psi) = \mathcal{M}_\Omega(U, v, \chi, \psi) + \int_\Omega \frac{|\nabla \rho|^2}{\rho^2} - \int_\Omega \nabla U \cdot \nabla \log \rho,$$

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$\mathcal{H}$ is the harmonic map energy for $\Phi = (u, v, \psi, \chi) \colon \mathbb{R}^3 \to \mathbb{H}^2_{\mathbb{C}}$. Since $\log\rho$ is harmonic, the EL equations for $\mathcal{H}_\Omega(u, v, \chi, \psi)$, and $\mathcal{M}_\Omega(u, v, \chi, \psi)$ are the same.

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where I_n is the interval of the z-axis Γ from p_n to p_{n+1}, $p_0 = -\infty$ and $p_{N+1} = \infty$.

Existence of the harmonic map

The tension of a map $\Phi = (u, v, \chi, \psi)$ is:

$$\begin{aligned}
\tau^u &= \Delta u - 2e^{4u}|\nabla v + \chi\nabla\psi - \psi\nabla\chi|^2 + e^{2u}\left(|\nabla\chi|^2 + |\nabla\psi|^2\right), \\
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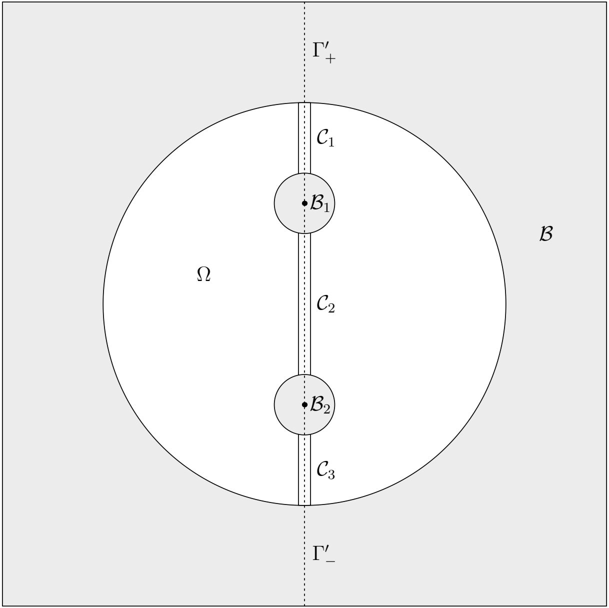
Lemma

There is a map $\tilde{\Phi}_0$, such that its reduced energy is finite, its tension $\tau(\tilde{\Phi}_0)$ has support inside a bounded set, and $\tau(\tilde{\Phi}_0)$ is pointwise bounded. Moreover, the values of $(\tilde{v}_0, \tilde{\chi}_0, \tilde{\psi}_0)$ agree with those of the given data (v, χ, ψ) on each component of $\Gamma' = \Gamma \setminus \{p_1, \dots, p_N\}$, and $\tilde{U}_0 = \tilde{u}_0 + \log\rho = \log r_n + o_{l-4}(r_n^{1/2})$ near each puncture p_n.


Γ'_+
$\mathcal{C}_1$
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$\mathcal{B}$
Ω
$\mathcal{C}_2$
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We say that two maps $\Phi, \Psi \colon \mathbb{R}^3 \setminus \Gamma \to \mathbb{H}^2_{\mathbb{C}}$ are *asymptotic* if $\mathrm{dist}_{\mathbb{H}^2_{\mathbb{C}}}(\Phi, \Psi)$ is bounded.

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Corollary

For any set of punctures p_n on the axis Γ and prescribed constants $v_0|_{I_n}$, $\chi_0|_{I_n}$, $\psi_0|_{I_n}$, $n = 1, \ldots, N$, there exists a corresponding unique harmonic map $\tilde{\Psi}_0 = (u_0, v_0, \chi_0, \psi_0)$ which is asymptotic to $\tilde{\Phi}_0$, and satisfies

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The proof of the corollary uses the convexity of $\mathrm{dist}_{\mathbb{H}^2_{\mathbb{C}}}$.

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Thank you!