

---

**Towards solution of negative mode problem  
in quantum tunnelling with gravity**

GR 21, July 14, 2016, NY

George Lavrelashvili

Department of Theoretical Physics

A.Razmadze Mathematical Institute

I.Javakhishvili Tbilisi State University

Tbilisi, Georgia

Supported by: the Shota Rustaveli NSF Grant No. FR/143/6-350/14

---

## Plan of the talk

- Introduction: Euclidean approach to metastable vacuum decay
- Negative mode problem in tunnelling transitions with gravity
- Historical Summary
- Concluding remarks

## Bounces and false vacuum decay

Coleman (1977)

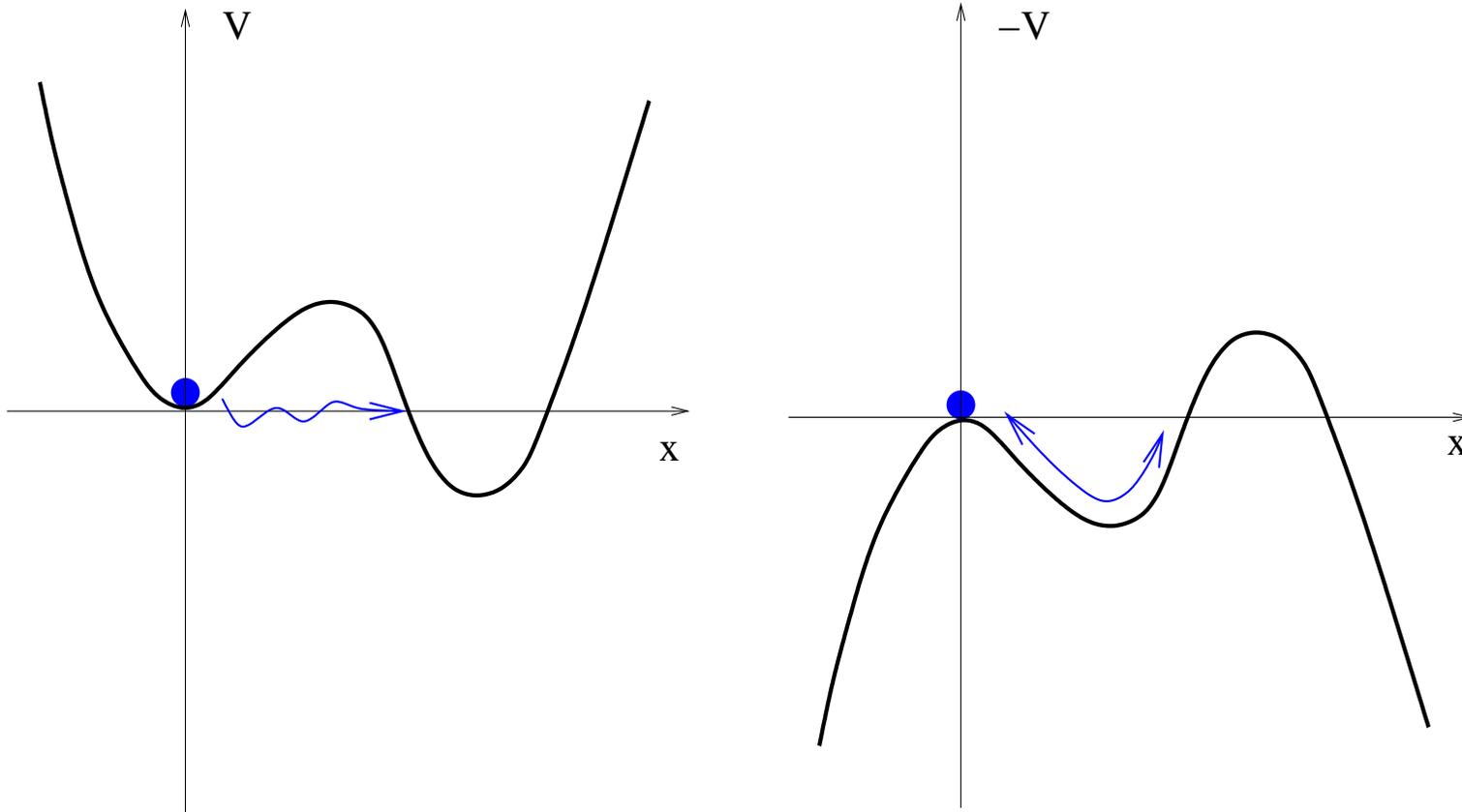


Figure 1: Tunneling in asymmetric double well potential.

## Euclidean approach to tunneling

Coleman (1977)

In the semiclassical approximation, summing multibounce configuration one finds correction to ground state energy  $E_0 = \hbar\omega/2$  in the following form

$$E = E_0 - \hbar K e^{-S/\hbar} [1 + O(\hbar)] , \quad (1)$$

where,  $S = \int d\eta [\frac{1}{2} (\frac{dx}{d\eta})^2 + V(x)]$  is the Euclidean action on the bounce solution  $\bar{x}(\eta)$  and pre-exponential factor  $K$  is given by (Gaussian) integration of the exponential of quadratic action of linear perturbations  $S^{(2)} = \frac{1}{2} \int d\eta \delta x [-\partial_\eta^2 + V''] \delta x$ ,

$$K = \frac{1}{2} \left( \frac{S}{2\pi\hbar} \right)^{1/2} \left( \frac{\det' [-\partial_\eta^2 + V''(\bar{x})]}{\det [-\partial_\eta^2 + \omega^2]} \right)^{-1/2} . \quad (2)$$

There is exactly one tunnelling negative mode in the spectrum of linear perturbations about the bounce solution, since (translational) zero energy wave function  $\psi_0 \sim \frac{d\bar{x}}{d\eta}$  of corresponding Schrödinger equation has a node. i.e.  $K = i\Gamma$ .

---

Finally, the decay probability per unit time of the unstable state is given by

$$\begin{aligned}\Gamma &= -2\text{Im}E/\hbar \\ &= \left(\frac{S}{2\pi\hbar}\right)^{1/2} \left| \frac{\det'[-\partial_\lambda^2 + V''(\bar{x})]}{\det[-\partial_\lambda^2 + \omega^2]} \right|^{-1/2} e^{-S/\hbar} [1 + O(\hbar)] .\end{aligned}\tag{3}$$

In the 1988 NPB article “Quantum Tunneling And Negative Eigenvalues,” Coleman arrives to strong conclusion: “There may exist solutions in other ways like bounces and which have more than one negative eigenvalue, but, even if they do exist, they have nothing to do with tunnelling.”

These quantum mechanical results could be generalized for

- Field theory in flat space-time [Frampton \(1976\)](#); [Coleman, Callan and Coleman \(1977\)](#)
- Field theory with gravity [Coleman and De Luccia \(1980\)](#)

## Special classical solutions in Euclidean quantum gravity

Hawking and Moss (1982)

1. The Hawking-Moss solution is a 4-sphere corresponding to scalar field sitting on the top of the potential barrier

$$\phi(\eta) = \phi_{top}, \quad \rho(\eta) = \mathcal{H}_{top}^{-1} \sin(\mathcal{H}_{top}\eta), \quad (4)$$

with  $\mathcal{H}_{top} = \sqrt{\kappa V(\phi_{top})/3}$ . Note that the Euclidean time  $\eta$  varies in finite interval  $\eta = (0, \eta_f)$ .

Coleman and De Luccia (1980)

2. The Coleman-De Luccia bounce is a deformed 4-sphere. It starts with some  $\phi = \phi_0$  at  $\eta = 0$  close to  $\phi_-$ , stops at  $\eta = \eta_f$  close to  $\phi_+$  and obeys the regularity conditions

$$\rho(0) = \dot{\phi}(0) = 0, \quad \rho(\eta_f) = \dot{\phi}(\eta_f) = 0. \quad (5)$$

Bousso and Linde (1998), Balek and Demetrian (2004)

Hackworth and Weinberg (2005)

B.-H. Lee, C. H. Lee, W. Lee and C. Oh (2010, 2012)

3. Oscillating bounces and instantons, solutions in which the scalar field passes over the barrier more than once.

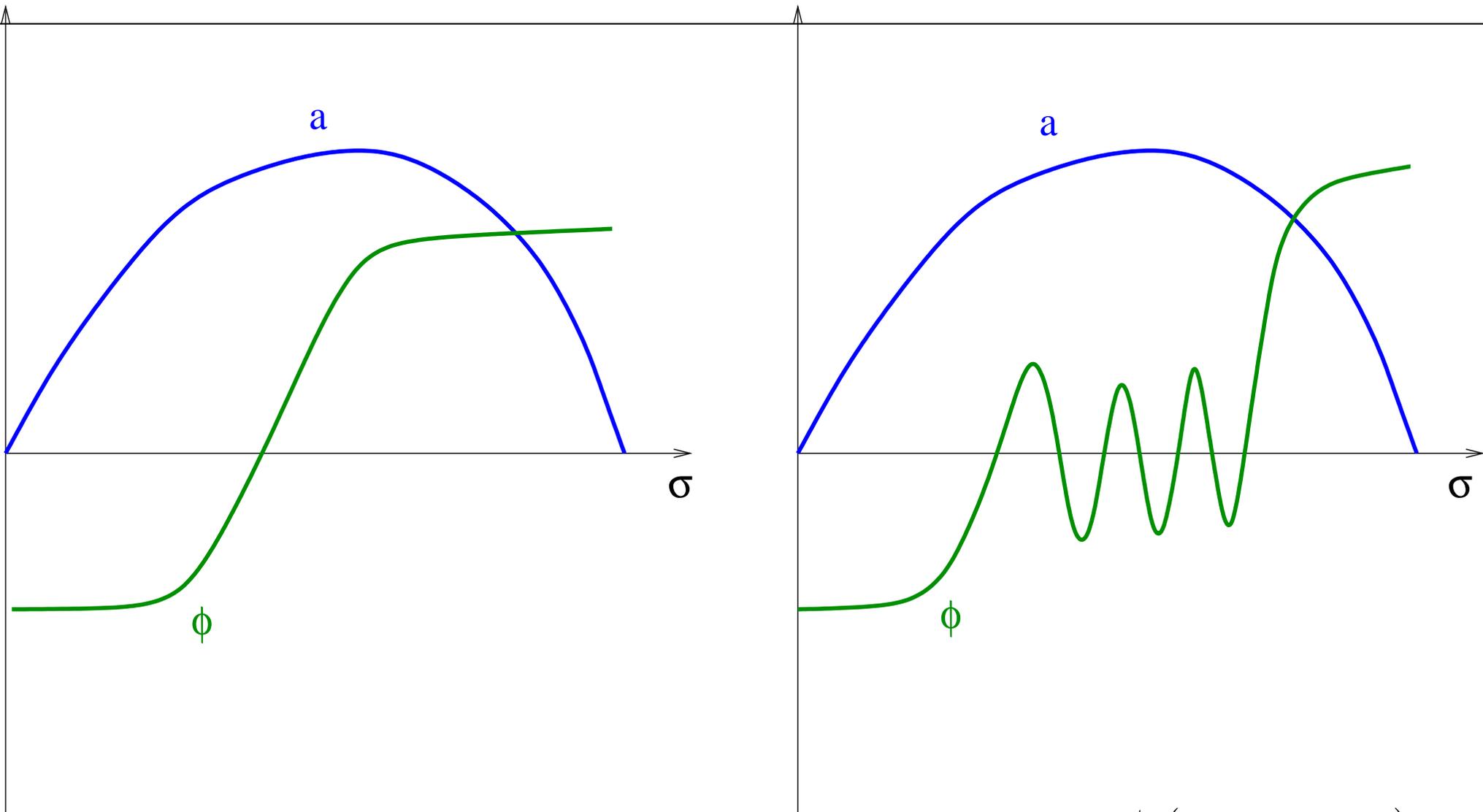


Figure 2: CDL bounce and oscillating bounce solution with  $N=7$  nodes of  $\phi$ , ( $\sigma \equiv \eta$ ,  $a \equiv \rho$ ).

## Negative mode problem

The Euclidean action of system composed of scalar field minimally coupled to gravity is

$$S = \int d^4x \sqrt{g} \left[ -\frac{1}{2\kappa} R + \frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi + V(\phi) \right], \quad (6)$$

where  $\kappa = 8\pi G$  is the reduced Newton's constant. For the  $O(4)$ -symmetric metric ansatz we will use

$$ds^2 = d\eta^2 + \rho^2(\eta) d\Omega_3^2 = a^2(\tau) (d\tau^2 + d\Omega_3^2), \quad (7)$$

where  $\eta$  is (Euclidean) proper time,  $\tau$ - conformal time,  $\rho(\eta)$ , (same as  $a(\tau)$ ), - is the scale factor and  $d\Omega_3^2$  is metric of unit three-sphere.

We expand the metric and the scalar field over a  $O(4)$ -symmetric background

$$\begin{aligned} ds^2 &= a(\tau)^2 \left[ (1 + 2A(\tau)) d\tau^2 + \gamma_{ij} (1 - 2\Psi(\tau)) dx^i dx^j \right], \\ \phi &= \varphi(\tau) + \Phi(\tau), \end{aligned} \quad (8)$$

where  $\tau$  is the conformal time,  $a$  and  $\varphi$  are the background field values and  $A$ ,  $\Psi$  and  $\Phi$  are small perturbations.  $\dot{\phantom{x}} \equiv d/d\eta$ ,  $' \equiv d/d\tau$ .

Expanding the total action to second order in perturbations and using the background equations of motion, we find

$$S = S^{(0)}[a, \varphi] + S^{(2)}[A, \Psi, \Phi] , \quad (9)$$

where  $S^{(0)}$  is the action of the background solution and  $S^{(2)}[A, \Psi, \Phi]$  is the quadratic action for scalar  $O(4)$ -symmetric perturbations given by the Lagrangian:

$$\begin{aligned} {}^{(s)}\mathcal{L} = & \frac{1}{2\kappa} a^2 \sqrt{\gamma} \left[ -6\Psi'^2 + 6\mathcal{K}\Psi^2 + \kappa(\Phi'^2 + a^2 \frac{\delta^2 V}{\delta\varphi\delta\varphi} \Phi^2 + 6\varphi'\Psi'\Phi) \right. \\ & \left. - (2\kappa\varphi'\Phi' - 2\kappa a^2 \frac{\delta V}{\delta\varphi} \Phi + 12\mathcal{H}\Psi' + 12\mathcal{K}\Psi)A - 2(\mathcal{H}' + 2\mathcal{H}^2 + \mathcal{K})A^2 \right] , \end{aligned}$$

with  $\mathcal{H} = a'/a$ ,  $\mathcal{K} = \pm 1, 0$ . Note that the variation with respect to  $A$  gives the first order (constraint) equation

$$2\kappa\varphi'\Phi' - 2\kappa a^2 \frac{\delta V}{\delta\varphi} \Phi + 12\mathcal{H}\Psi' + 12\mathcal{K}\Psi + 4(\mathcal{H}' + 2\mathcal{H}^2 + \mathcal{K})A = 0 .$$

To obtain unconstrained (physical) degree of freedom one should impose gauge condition and solve constraints.

In frame of the Hamiltonian approach, fixing the gauge by  $A = 0$ ,  $\Pi_\Psi = 0$  one obtains unconstrained quadratic action for one physical dynamical degree of freedom for  $\mathcal{K} = +1$  as

$$S_E^{(2)}[\Phi] = 2\pi^2 \int \rho^3(\eta) d\eta \left[ \frac{1}{2Q(\eta)} \dot{\Phi}^2 + \frac{1}{2} U[\varphi(\eta), \rho(\eta)] \Phi^2 \right], \quad (10)$$

where the factor  $Q$  was given by

$$Q := 1 - \frac{\kappa \rho^2 \dot{\varphi}^2}{6}, \quad (11)$$

and the potential  $U$  is expressed in terms of the bounce solution as

$$U[\varphi(\eta), \rho(\eta)] \equiv \frac{V_{,\varphi\varphi}(\varphi)}{Q} + \frac{2\kappa \dot{\varphi}^2}{Q} + \frac{\kappa}{3Q^2} \left( 6\dot{\rho}^2 \dot{\varphi}^2 + \rho^2 V_{,\varphi}^2(\varphi) - 5\rho \dot{\rho} \dot{\varphi} V_{,\varphi}(\varphi) \right). \quad (12)$$

The exact form of the fluctuation operator depends on the choice of a weight function, which can be specified by defining a norm.

With the natural choice of the norm

$$\|\Phi\|^2 \equiv \int d^4x \sqrt{g} \Phi^2 = 2\pi^2 \int d\eta \rho(\eta)^3 \Phi^2 . \quad (13)$$

The fluctuation equation diagonalizing the quadratic action Eq. (10) then has the form

$$-\frac{1}{Q} \frac{d^2\Phi}{d\eta^2} + \left( \frac{\dot{Q}}{Q^2} - \frac{3\dot{\rho}}{\rho Q} \right) \frac{d\Phi}{d\eta} + U\Phi = \lambda\Phi , \quad (14)$$

with the potential  $U$  given in Eq. (12). Note that the function  $Q \rightarrow 1$  at the ends of the interval  $[0, \eta_{max}]$ , but for some bounces it can become negative for some interval of  $\eta$ .

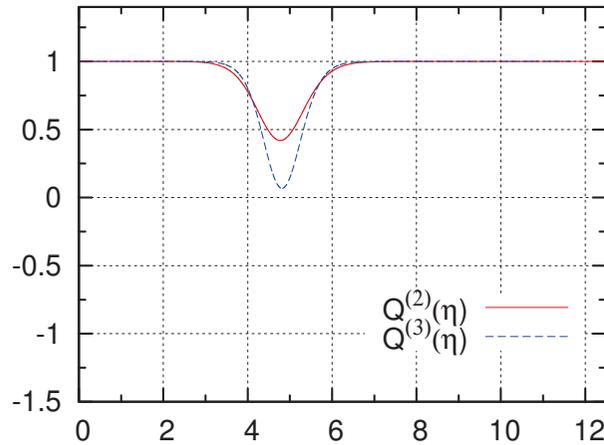
Main results:

1. If  $Q > 0$ , one finds exactly one tunnelling negative mode for all bounces
2. If  $Q$  becomes negative in some interval
  - a). Solution of pulsation equation and first derivative is regular across  $Q = 0$  points.
  - b). On top of one "tunneling" negative mode, an infinite tower of additional negative modes arise with support in  $Q < 0$  region.

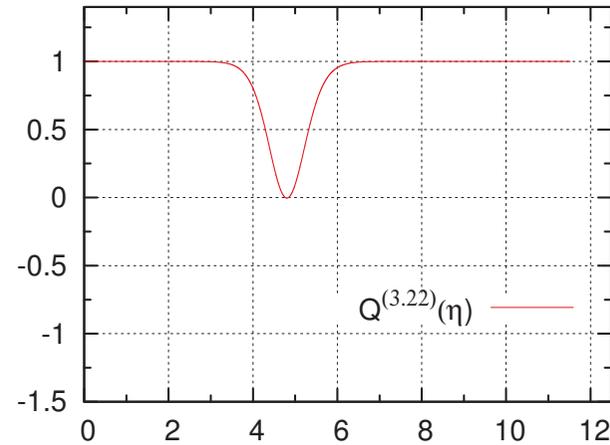
Example potential used by Lee & Weinberg (2014)

$$V(\varphi) = B(\varphi^2 - 0.25)^2 + 0.1(\varphi + 1), \quad (15)$$

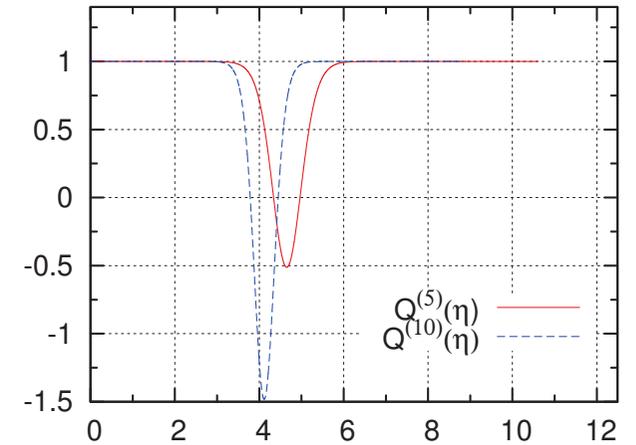
and corresponding factor  $Q$  for different values of parameter  $B$ :



(a)  $B = 2$  and  $B = 3$

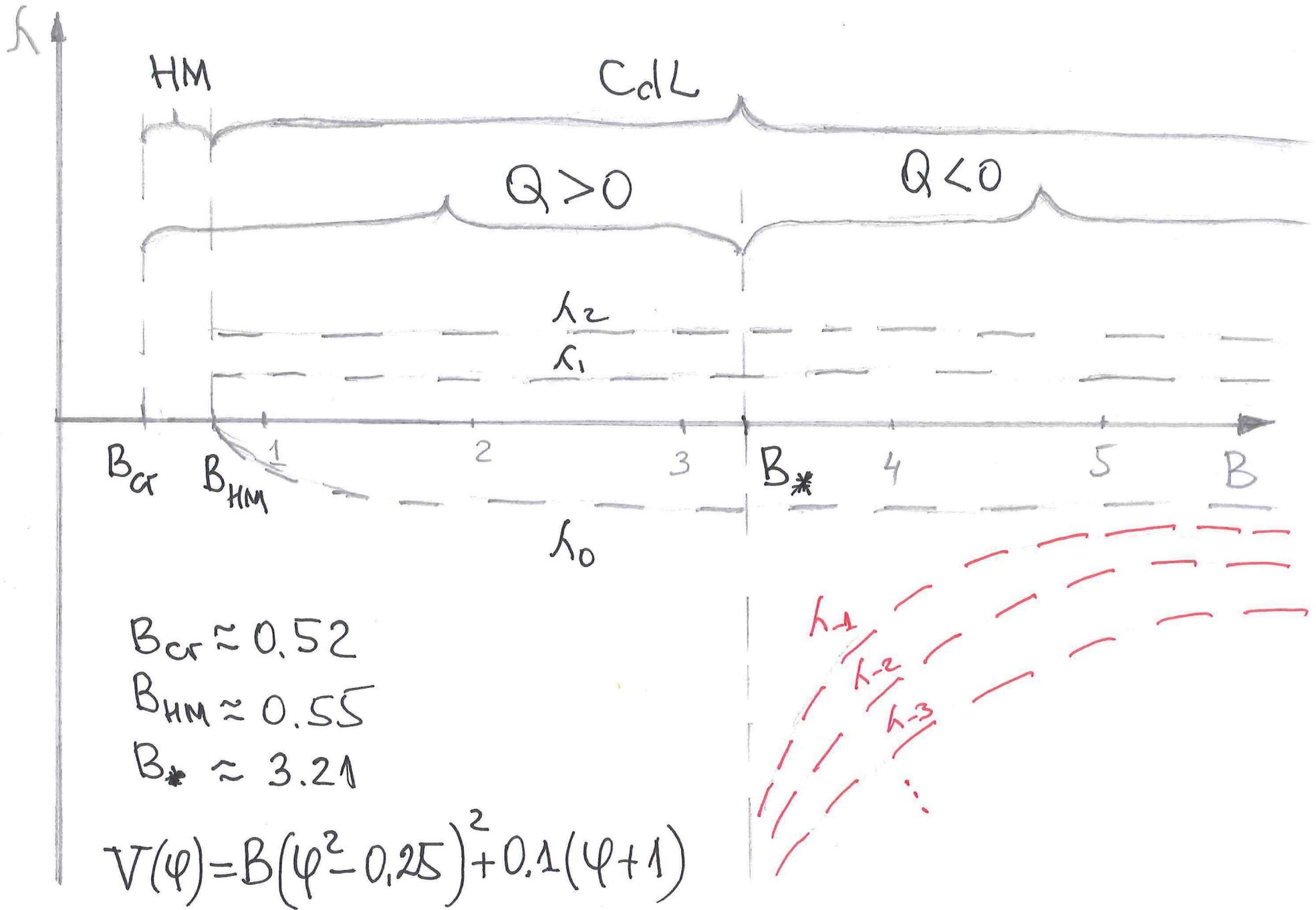


(b)  $B = 3.22$  ( $Q < 0$  briefly)



(c)  $B = 5$  and  $B = 10$

Figure 3: The kinetic pre-factor  $Q(\eta)$  for the background potential (15) and different values of  $B$ . For  $B \approx 3.22$  the factor  $Q$  becomes negative along some interval of Euclidean time  $\eta$ .



Working in Lagrangian approach with gauge invariant variable  $\chi$ ,

$$\chi \equiv \dot{\rho}\Phi - \rho\dot{\phi}\Psi \quad \text{and factor} \quad Q_{LRT} := \dot{\rho}^2 - \frac{\kappa\rho^2\dot{\phi}^2}{6} = 1 - \frac{\kappa\rho^2}{3}V, \quad (16)$$

Lee and Weinberg solved numerically with Mathematica pulsation equation for concrete potentials and arrived to conclusion that type A bounces have tunneling negative mode whereas type B bounces don't!

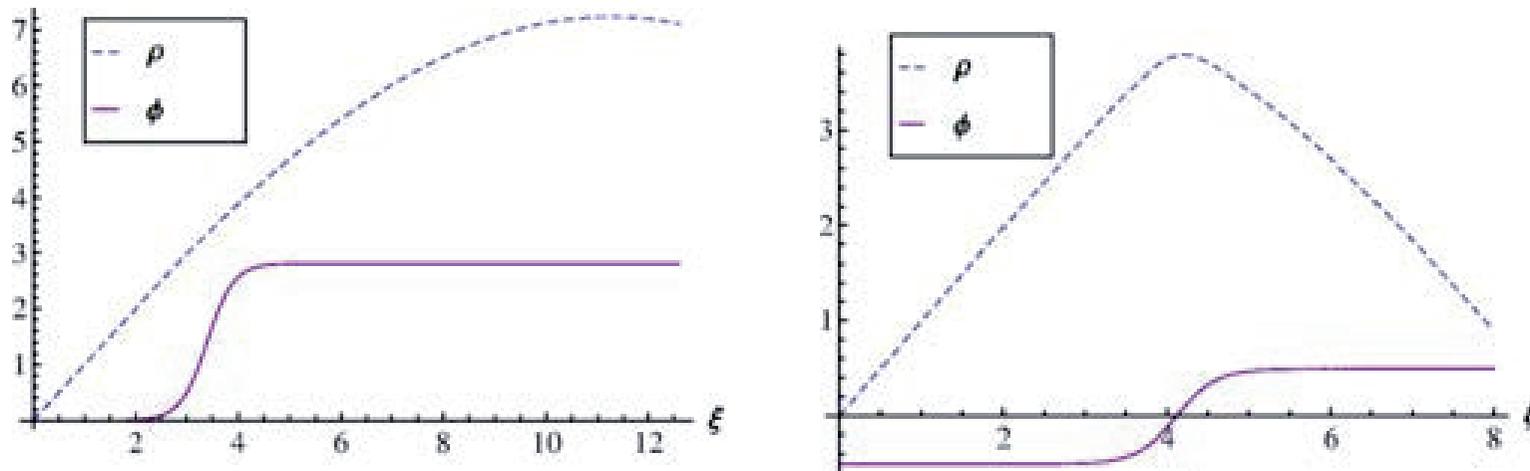


Figure 4: Type A bounces (left panel) and type B (right panel).

## Historical Summary of the negative mode problem

Year / Authors	Tunneling negative mode	Additional negative modes
1985, G.L., Rubakov Tinyakov	Not discussed	Infinitely many
1992, Tanaka, Sasaki	None	None
2000, Khvedelidze, G.L., Tanaka	One	Not discussed (only $Q > 0$ case)
2001, Gratton, Turok	One	Not discussed (only $Q > 0$ case)
2006, G.L.	N for the N-th oscillating bounce	Not discussed (only $Q > 0$ case)
2014, Lee, Weinberg	One for type A / None for type B	Infinitely many
2015, Koehn, G.L., Lehnert	One	Infinitely many in $Q < 0$ case

Tab 1. Conclusion about the number of negative modes reached in different investigations.

---

## Concluding remarks

1. We have found that with the proper reduction scheme CdL solution has exactly one negative mode for  $Q > 0$  backgrounds, as it should be for proper bounce.
2. The oscillating instantons and bounces with  $N$  nodes have exactly  $N$  homogeneous negative modes in their spectrum of linear perturbations. Existence of more than one negative modes makes obscure the relation of these oscillating bounce solutions to the false vacuum decay processes.
3. When  $Q < 0$  along the bounce:
  - a). Solution of pulsation equation is smooth across  $Q = 0$  points.
  - b). Single tunneling negative mode continues to exist.
  - c). Additional, infinite number of negative modes appears.

### Challenge:

- How to interpret an infinite tower of additional negative modes for  $Q < 0$  cases: Their existence and significance remain mysterious even after 30 years.

---

Thank you for attention!