Kinematic restrictions on particle collisions near extremal black holes – a unified analysis

Filip Hejda

Institute of Theoretical Physics, Faculty of Mathematics and Physics,
Charles University in Prague

with Jiří Bičák...

Kinematic restrictions on particle collisions near extremal black holes – a unified analysis

Filip Hejda

Institute of Theoretical Physics, Faculty of Mathematics and Physics,
Charles University in Prague
Centro Multidisciplinar de Astrofísica, Departamento de Física, Instituto Superior Técnico,
Universidade de Lisboa

with Jiří Bičák…
… and J. P. S. Lemos

TÉCNICO LISBOA

PD F PROGRAMAS DE DOUTORAMENTO FCT

History of black holes as particle colliders

- R. Penrose, *Gravitational Collapse: the Role of General Relativity*, Rivista del Nuovo Cimento, Numero Speziale, 252 (1969).

History of black holes as particle colliders

- R. Penrose, *Gravitational Collapse: the Role of General Relativity*, Rivista del Nuovo Cimento, Numero Speziale, 252 (1969).
- J. Bardeen, W. H. Press, S. A. Teukolsky, *Rotating black holes: Locally nonrotating frames, energy extraction, and scalar synchrotron radiation*, The Astrophysical Journal **178**, 347-370 (1972).

History of black holes as particle colliders

- R. Penrose, *Gravitational Collapse: the Role of General Relativity*, Rivista del Nuovo Cimento, Numero Speziale, 252 (1969).
- J. Bardeen, W. H. Press, S. A. Teukolsky, *Rotating black holes: Locally nonrotating frames, energy extraction, and scalar synchrotron radiation*, The Astrophysical Journal **178**, 347-370 (1972).
- T. Piran, J. Shaham, J. Katz, *High efficiency of the Penrose mechanism for particle collisions*, The Astrophysical Journal Letters **196**, L107-L108 (1975).

History of black holes as particle colliders

- R. Penrose, *Gravitational Collapse: the Role of General Relativity*, Rivista del Nuovo Cimento, Numero Speziale, 252 (1969).
- J. Bardeen, W. H. Press, S. A. Teukolsky, *Rotating black holes: Locally nonrotating frames, energy extraction, and scalar synchrotron radiation*, The Astrophysical Journal **178**, 347-370 (1972).
- T. Piran, J. Shaham, J. Katz, *High efficiency of the Penrose mechanism for particle collisions*, The Astrophysical Journal Letters **196**, L107-L108 (1975).
- M. Bañados, J. Silk, S. M. West, *Kerr Black Holes as Particle Accelerators to Arbitrarily High Energy*, Physical Review Letters **103**, 111102 (2009).

History of black holes as particle colliders

- R. Penrose, *Gravitational Collapse: the Role of General Relativity*, Rivista del Nuovo Cimento, Numero Speziale, 252 (1969).
- J. Bardeen, W. H. Press, S. A. Teukolsky, *Rotating black holes: Locally nonrotating frames, energy extraction, and scalar synchrotron radiation*, The Astrophysical Journal **178**, 347-370 (1972).
- T. Piran, J. Shaham, J. Katz, *High efficiency of the Penrose mechanism for particle collisions*, The Astrophysical Journal Letters **196**, L107-L108 (1975).
- M. Bañados, J. Silk, S. M. West, *Kerr Black Holes as Particle Accelerators to Arbitrarily High Energy*, Physical Review Letters **103**, 111102 (2009).
- … and then dozens of others…

Canonical review

- General axially symmetric stationary metric as a model of an isolated black hole

$$\boldsymbol{g} = -N^2\,\mathbf{d}t^2 + g_{\varphi\varphi}\,(\mathbf{d}\varphi - \omega\,\mathbf{d}t)^2 + g_{rr}\,\mathbf{d}r^2 + g_{\vartheta\vartheta}\,\mathbf{d}\vartheta^2$$

- Electromagnetic potential and generalised electrostatic potential ϕ

$$\boldsymbol{A} = A_t\,\mathbf{d}t + A_\varphi\,\mathbf{d}\varphi = -\phi\,\mathbf{d}t + A_\varphi\,(\mathbf{d}\varphi - \omega\,\mathbf{d}t)$$

- Canonical formalism (analogous to classical mechanics) for motion of a test particle with charge $q = m\tilde{q}$

$$\mathscr{L} = \frac{1}{2} m g_{\mu\nu} u^\mu u^\nu + qA_\mu u^\mu \qquad \mathit{\Pi}_\alpha = \frac{\partial \mathscr{L}}{\partial u^\alpha} = p_\alpha + qA_\alpha$$

Canonical review

- General axially symmetric stationary metric as a model of an isolated black hole

$$\boldsymbol{g} = -N^2 \, \mathbf{d}t^2 + g_{\varphi\varphi} \left(\mathbf{d}\varphi - \omega \, \mathbf{d}t\right)^2 + g_{rr} \, \mathbf{d}r^2 + g_{\vartheta\vartheta} \, \mathbf{d}\vartheta^2$$

- Electromagnetic potential and generalised electrostatic potential ϕ

$$\boldsymbol{A} = A_t \, \mathbf{d}t + A_\varphi \, \mathbf{d}\varphi = -\phi \, \mathbf{d}t + A_\varphi \left(\mathbf{d}\varphi - \omega \, \mathbf{d}t\right)$$

- Canonical formalism (analogous to classical mechanics) for motion of a test particle with charge $q = m\tilde{q}$

$$\mathscr{L} = \frac{1}{2} m g_{\mu\nu} u^\mu u^\nu + q A_\mu u^\mu \qquad \mathit{\Pi}_\alpha = \frac{\partial \mathscr{L}}{\partial u^\alpha} = p_\alpha + q A_\alpha$$

- Constants of motion and reduced (per unit rest mass) constants of motion (ε, l) of a particle with rest mass m

$$-\mathit{\Pi}_t = -p_t - q A_t = E = \varepsilon m$$
$$\mathit{\Pi}_\varphi = p_\varphi + q A_\varphi = L_z = l m$$

First-order equations of motion

- Two constants of motion give two first-order equations of motion

$$u^t = \frac{\mathscr{X}}{N^2} \qquad\qquad u^\varphi = \frac{\omega \mathscr{X}}{N^2} + \frac{l - \tilde{q} A_\varphi}{g_{\varphi\varphi}}$$

- "Forwardness"

$$\mathscr{X} \equiv \varepsilon - \omega l - \tilde{q}\phi > 0$$

First-order equations of motion

- Two constants of motion give two first-order equations of motion

$$u^t = \frac{\mathscr{X}}{N^2} \qquad\qquad u^\varphi = \frac{\omega \mathscr{X}}{N^2} + \frac{l - \tilde{q} A_\varphi}{g_{\varphi\varphi}}$$

- "Forwardness"

$$\mathscr{X} \equiv \varepsilon - \omega l - \tilde{q}\phi > 0$$

- Assuming "mirror symmetry" and initial conditions $\vartheta = \pi/2$, $u^\vartheta = 0$, we can get the third equation from velocity normalisation

$$u^r = \pm\sqrt{\frac{1}{N^2 g_{rr}}\left[\mathscr{X}^2 - N^2\left(1 + \frac{(l - \tilde{q}A_\varphi)^2}{g_{\varphi\varphi}}\right)\right]}$$

- Effective potential ($\varepsilon \geqslant V$)

$$V = \omega l + \tilde{q}\phi + N\sqrt{1 + \frac{(l - \tilde{q}A_\varphi)^2}{g_{\varphi\varphi}}}$$

Centre-of-mass collision energy

- In test particle approximation, let us consider two colliding particles with rest masses m_1, m_2 and velocities $u^\mu_{(1)}, u^\mu_{(2)}$
- In a suitable frame connected with the instant of collision it holds

$$(E_{\mathrm{CM}}, 0, 0, 0) = m_1 \boldsymbol{u}_{(1)} + m_2 \boldsymbol{u}_{(2)}$$

Centre-of-mass collision energy

- In test particle approximation, let us consider two colliding particles with rest masses m_1, m_2 and velocities $u^{\mu}_{(1)}, u^{\mu}_{(2)}$
- In a suitable frame connected with the instant of collision it holds

$$(E_{\mathrm{CM}}, 0, 0, 0) = m_1 \boldsymbol{u}_{(1)} + m_2 \boldsymbol{u}_{(2)}$$

- We can "take a square" of this expression

$$\frac{E^2_{\mathrm{CM}}}{2m_1 m_2} = \frac{m_1}{2m_2} + \frac{m_2}{2m_1} - g_{\mu\nu} u^{\mu}_{(1)} u^{\nu}_{(2)}$$

- This defines an invariant called „Centre-of-mass collision energy"
- One can investigate its behaviour depending on the selected curved background, parameters of the particles and the point of collision

Collision energy on the horizon and the critical condition

- We can plug the first order EOM into the collision energy formula
- Let us take the limit $N \to 0$

$$\left.\frac{E_{\mathrm{CM}}^2}{2m_1 m_2}\right|_{N=0} = \frac{m_1}{2m_2} + \frac{m_2}{2m_1} - \left.\frac{\left(l_1 - \tilde{q}_1 A_\varphi\right)\left(l_2 - \tilde{q}_2 A_\varphi\right)}{g_{\varphi\varphi}}\right|_{N=0} +$$
$$+ \frac{1}{2}\left.\left[1 + \frac{\left(l_1 - \tilde{q}_2 A_\varphi\right)^2}{g_{\varphi\varphi}}\right]\right|_{N=0} \frac{\mathscr{X}_1^{\mathrm{H}}}{\mathscr{X}_2^{\mathrm{H}}} + \frac{1}{2}\left.\left[1 + \frac{\left(l_1 - \tilde{q}_1 A_\varphi\right)^2}{g_{\varphi\varphi}}\right]\right|_{N=0} \frac{\mathscr{X}_2^{\mathrm{H}}}{\mathscr{X}_1^{\mathrm{H}}}$$

Collision energy on the horizon and the critical condition

- We can plug the first order EOM into the collision energy formula
- Let us take the limit $N \to 0$

$$\left.\frac{E_{\text{CM}}^2}{2m_1 m_2}\right|_{N=0} = \frac{m_1}{2m_2} + \frac{m_2}{2m_1} - \left.\frac{\left(l_1 - \tilde{q}_1 A_\varphi\right)\left(l_2 - \tilde{q}_2 A_\varphi\right)}{g_{\varphi\varphi}}\right|_{N=0} +$$
$$+ \frac{1}{2}\left.\left[1 + \frac{\left(l_1 - \tilde{q}_2 A_\varphi\right)^2}{g_{\varphi\varphi}}\right]\right|_{N=0} \frac{\mathscr{X}_1^{\text{H}}}{\mathscr{X}_2^{\text{H}}} + \frac{1}{2}\left.\left[1 + \frac{\left(l_1 - \tilde{q}_1 A_\varphi\right)^2}{g_{\varphi\varphi}}\right]\right|_{N=0} \frac{\mathscr{X}_2^{\text{H}}}{\mathscr{X}_1^{\text{H}}}$$

- The limit is unbounded for so-called critical particle, with $\mathscr{X}^{\text{H}} = 0$
- This can be understood as a restriction for energy

$$\varepsilon_{\text{cr}} = l\omega_{\text{H}} + \tilde{q}\phi_{\text{H}} = V\big|_{r=r_+}$$

Collision energy on the horizon and the critical condition

- We can plug the first order EOM into the collision energy formula
- Let us take the limit $N \to 0$

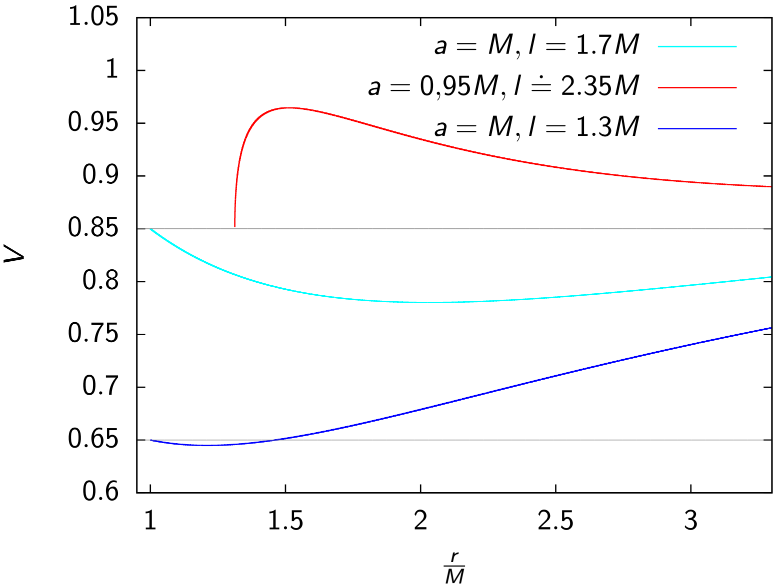
$$\left.\frac{E_{\mathrm{CM}}^2}{2m_1m_2}\right|_{N=0} = \frac{m_1}{2m_2} + \frac{m_2}{2m_1} - \left.\frac{\left(l_1 - \tilde{q}_1 A_\varphi\right)\left(l_2 - \tilde{q}_2 A_\varphi\right)}{g_{\varphi\varphi}}\right|_{N=0} + $$
$$+ \frac{1}{2}\left.\left[1 + \frac{\left(l_1 - \tilde{q}_2 A_\varphi\right)^2}{g_{\varphi\varphi}}\right]\right|_{N=0} \frac{\mathscr{X}_1^{\mathrm{H}}}{\mathscr{X}_2^{\mathrm{H}}} + \frac{1}{2}\left.\left[1 + \frac{\left(l_1 - \tilde{q}_1 A_\varphi\right)^2}{g_{\varphi\varphi}}\right]\right|_{N=0} \frac{\mathscr{X}_2^{\mathrm{H}}}{\mathscr{X}_1^{\mathrm{H}}}$$

- The limit is unbounded for so-called critical particle, with $\mathscr{X}^{\mathrm{H}} = 0$
- This can be understood as a restriction for energy

$$\varepsilon_{\mathrm{cr}} = l\omega_{\mathrm{H}} + \tilde{q}\phi_{\mathrm{H}} = \left.V\right|_{r=r_+}$$

- O. B. Zaslavskii, *Acceleration of particles as a universal property of rotating black holes*, Physical Review D **82**, 083004 (2010).
- O. B. Zaslavskii, *Acceleration of particles by nonrotating charged black holes?* JETP Letters **92**, 571-574 (2011).

Curves of the effective potential for Kerr black holes



When can critical particles reach the collision point?

- As $\varepsilon_{\mathrm{cr}} = V|_{r_+}$, the effective potential must decrease in order for the motion of critical particles towards r_+ to be allowed: look at ${}^{\partial V}\!/_{\partial r}|_{r_+}$
- In the subextremal case the derivative is always infinite and positive, so no critical particle can approach r_+

When can critical particles reach the collision point?

- As $\varepsilon_{\mathrm{cr}} = V|_{r_+}$, the effective potential must decrease in order for the motion of critical particles towards r_+ to be allowed: look at ${}^{\partial V}/{}_{\partial r}|_{r_+}$
- In the subextremal case the derivative is always infinite and positive, so no critical particle can approach r_+
- For extremal black holes the derivative is finite and can have either sign, depending on the BH model and the particle parameters $\tilde{q}, l$
- Let us treat the condition

$$\left.\frac{\partial V}{\partial r}\right|_{r=r_0} = 0$$

as a prescription of a curve in variables $\tilde{q}, l$

When can critical particles reach the collision point?

- As $\varepsilon_{\mathrm{cr}} = V|_{r_+}$, the effective potential must decrease in order for the motion of critical particles towards r_+ to be allowed: look at ${}^{\partial V}/_{\partial r}|_{r_+}$
- In the subextremal case the derivative is always infinite and positive, so no critical particle can approach r_+
- For extremal black holes the derivative is finite and can have either sign, depending on the BH model and the particle parameters $\tilde{q}, l$
- Let us treat the condition

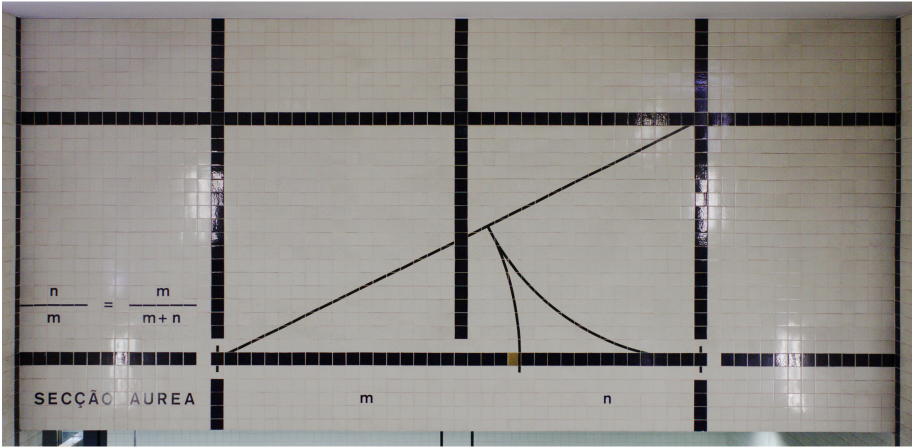
$$\left.\frac{\partial V}{\partial r}\right|_{r=r_0} = 0$$

as a prescription of a curve in variables $\tilde{q}, l$

- It is just a branch of the following hyperbola:

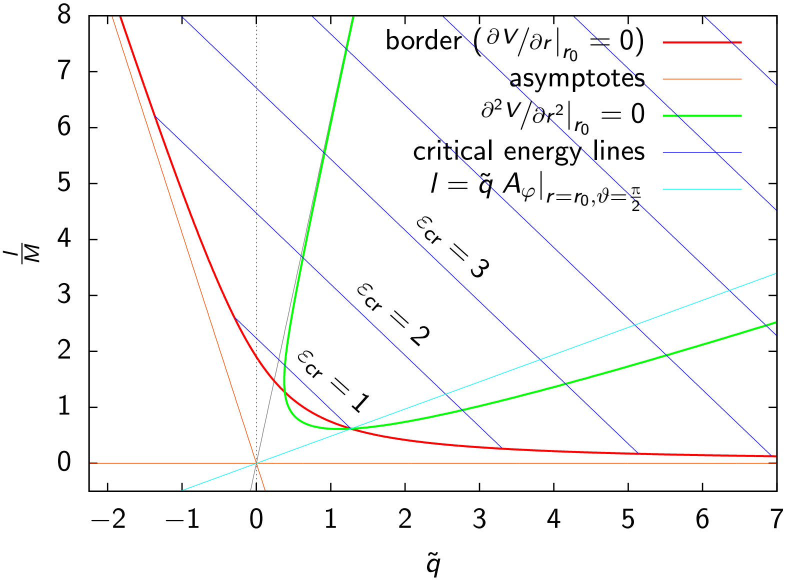
$$\left.\left\{ l^2 \left[\frac{\left(\frac{\partial \omega}{\partial r}\right)^2}{\tilde{N}^2} - \frac{1}{g_{\varphi\varphi}} \right] + \tilde{q}^2 \left[\frac{\left(\frac{\partial \phi}{\partial r}\right)^2}{\tilde{N}^2} - \frac{A_\varphi^2}{g_{\varphi\varphi}} \right] + 2l\tilde{q} \left(\frac{\frac{\partial \omega}{\partial r}\frac{\partial \phi}{\partial r}}{\tilde{N}^2} + \frac{A_\varphi}{g_{\varphi\varphi}} \right) \right\}\right|_{r=r_0, \vartheta=\frac{\pi}{2}} = 1$$

"Golden black hole"

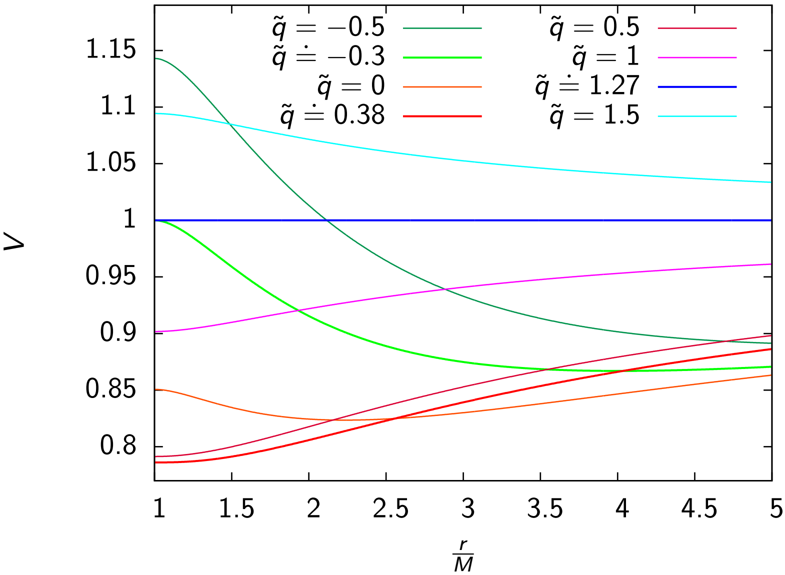


Secção áurea, Estação Saldanha, Metropolitano de Lisboa

"Golden black hole": $\frac{M^2}{Q^2} = \frac{Q^2}{a^2} = \frac{M}{a} = \frac{\sqrt{5}+1}{2}$



Curves of the effective potential for “Golden black hole”



What about the second derivative?

- When $\left.{}^{\partial V}\!/_{\partial r}\right|_{r_0}$ is small (or zero), the second order of Taylor expansion can overweigh it easily
- Thus, let us treat the condition

$$\left.\frac{\partial^2 V}{\partial r^2}\right|_{r=r_0} = 0$$

as a prescription of another curve in $\tilde{q}, l$ and look for intersections…

What about the second derivative?

- When $\left.{}^{\partial V}\!/\!{}_{\partial r}\right|_{r_0}$ is small (or zero), the second order of Taylor expansion can overweigh it easily
- Thus, let us treat the condition
$$\left.\frac{\partial^2 V}{\partial r^2}\right|_{r=r_0} = 0$$
as a prescription of another curve in $\tilde{q}, l$ and look for intersections...
- However, this one is more complicated than a hyperbola branch...

$$l = -\left.\left[\frac{\lambda\frac{\partial^2\phi}{\partial r^2}\sqrt{1+\frac{\lambda^2 A_\varphi^2}{g_{\varphi\varphi}}} + 2\frac{\partial\tilde{N}}{\partial r} + \left(2\frac{\partial\tilde{N}}{\partial r} - \frac{\tilde{N}}{g_{\varphi\varphi}}\frac{\partial g_{\varphi\varphi}}{\partial r} + 2\frac{\tilde{N}}{A_\varphi}\frac{\partial A_\varphi}{\partial r}\right)\frac{\lambda^2 A_\varphi^2}{g_{\varphi\varphi}}}{\left(\frac{\partial^2\omega}{\partial r^2}A_\varphi + \frac{\partial^2\phi}{\partial r^2}\right)\sqrt{1+\frac{\lambda^2 A_\varphi^2}{g_{\varphi\varphi}}} + 2\tilde{N}\frac{\lambda A_\varphi}{g_{\varphi\varphi}}\frac{\partial A_\varphi}{\partial r}} A_\varphi\right]\right|_{r=r_0,\vartheta=\frac{\pi}{2}}$$

$$\tilde{q} = \left.\frac{\lambda\frac{\partial^2\omega}{\partial r^2}A_\varphi\sqrt{1+\frac{\lambda^2 A_\varphi^2}{g_{\varphi\varphi}}} - 2\frac{\partial\tilde{N}}{\partial r} - \left(2\frac{\partial\tilde{N}}{\partial r} - \frac{\tilde{N}}{g_{\varphi\varphi}}\frac{\partial g_{\varphi\varphi}}{\partial r}\right)\frac{\lambda^2 A_\varphi^2}{g_{\varphi\varphi}}}{\left(\frac{\partial^2\omega}{\partial r^2}A_\varphi + \frac{\partial^2\phi}{\partial r^2}\right)\sqrt{1+\frac{\lambda^2 A_\varphi^2}{g_{\varphi\varphi}}} + 2\tilde{N}\frac{\lambda A_\varphi}{g_{\varphi\varphi}}\frac{\partial A_\varphi}{\partial r}}\right|_{r=r_0,\vartheta=\frac{\pi}{2}}$$

Concluding remarks

- Research article (almost) complete
- Possible follow up: include strong external magnetic fields, utilising the results of:

  J. Bičák, FH, *Near-horizon description of extremal magnetized stationary black holes and Meissner effect*, Phys. Rev. D **92**, 104006 (2015).
- These are closely related to the near-horizon limiting map and non-uniqueness of its inverse discussed yesterday by J. Lucietti…

Concluding remarks + Acknowledgements

- Research article (almost) complete
- Possible follow up: include strong external magnetic fields, utilising the results of:
  J. Bičák, FH, *Near-horizon description of extremal magnetized stationary black holes and Meissner effect*, Phys. Rev. D **92**, 104006 (2015).
- These are closely related to the near-horizon limiting map and non-uniqueness of its inverse discussed yesterday by J. Lucietti…

The work is/has been supported by:

- GAUK (Charles University in Prague) grant No. 196516 "Black Holes and Related Spacetimes in Four and Higher Dimensions"
- FCT (Portugal) grant Bolsa de Investigação, reference PD/BD/113477/2015
- GAUK (Charles University in Prague) grant No. 606412
- GAČR (Czech republic) grant No. 14-37086G