A Toy Penrose Inequality and Its Proof

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The Penrose Inequality

$$M \geq \sqrt{\frac{A}{16\pi}}$$

M : mass

A : area of surface

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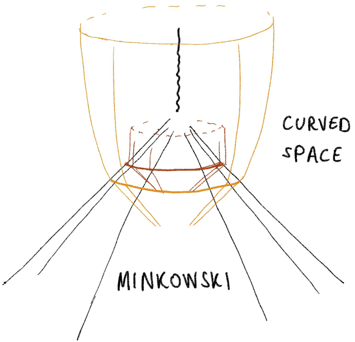

Null shell version

Penrose (1973), Gibbons (1973), Tod (1985), Mars & Soria (2016)

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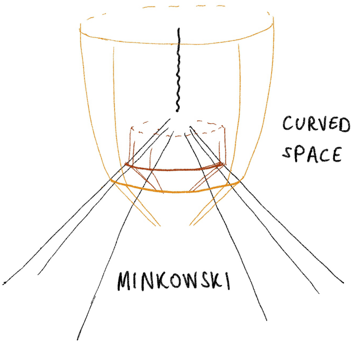

Null shell version

A geometric property of surfaces in Minkowski space:

$$\oint \theta_+ \, \mathrm{d}A \geq \sqrt{16\pi A}$$

θ_+ : outer null expansion of surface

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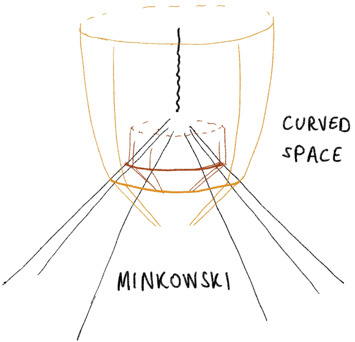
Saturated by the event horizon of the Schwarzschild black hole.



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The Toy Version

The nonrotating BTZ black hole:

$$\ell^2 M = \left(\frac{L_{BTZ}}{2\pi}\right)^2$$

M : mass of black hole

ℓ : length scale of adS_3

L_{BTZ} : length of event horizon

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Circular cross section of past light cone:

$$\frac{1}{2\pi}\oint \theta_+ \mathrm{d}l = \ell^2 + \left(\frac{L}{2\pi}\right)^2$$

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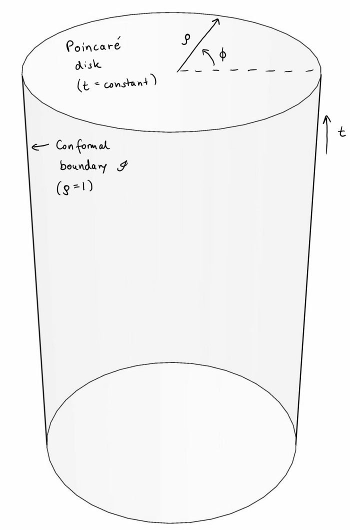
Toy version Penrose inequality in 2+1-dimensional anti-de Sitter space:

$$\boxed{\frac{1}{2\pi}\oint \theta_+ \, \mathrm{d}l \geq \ell^2 + \left(\frac{L}{2\pi}\right)^2}$$

The Proof



2+1-dimensional anti-de Sitter space as a stack of Poincaré disks:

$$\mathrm{d}s^2 = -\,\ell^2 \left(\frac{1+\rho^2}{1-\rho^2}\right)^2 \mathrm{d}t^2 + \frac{4\ell^2}{(1-\rho^2)^2}\left(\mathrm{d}\rho^2 + \rho^2 \mathrm{d}\phi^2\right)$$

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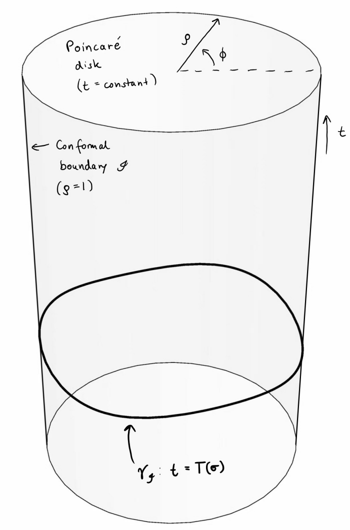

Null surface Σ

Characterized by its intersection with $\mathscr{I}$.

$$\gamma_{\mathscr{I}}: \quad \rho = 1, \quad \phi = \sigma, \quad t = T(\sigma)$$

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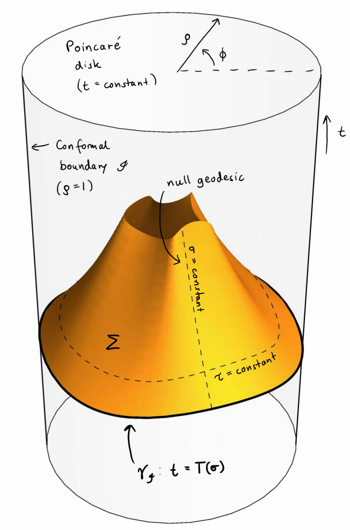

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Ruled by null geodesics parameterized by τ.

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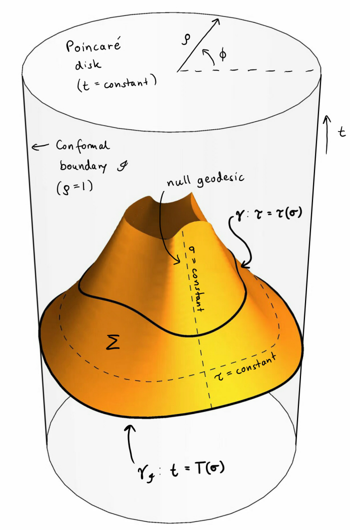

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$$\gamma : \quad \tau = \tau(\sigma) \quad \text{on } \Sigma$$

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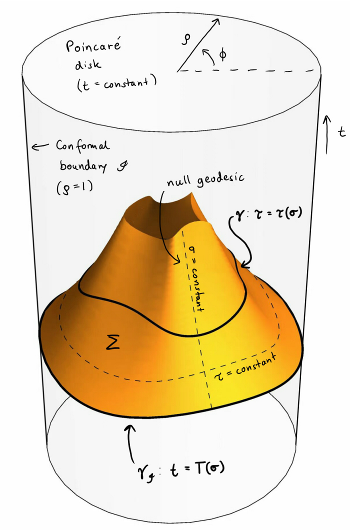

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Length: $L = \ell \int_0^{2\pi} \cot[\tau(\sigma)]\, d\sigma$

The Proof

Proof.

Once θ_+ has been calculated we observe that

$$\frac{1}{2\pi}\oint_\gamma \theta_+ \,\mathrm{d}l = \ell^2\left(1+\int_0^{2\pi}\cot^2[\tau(\sigma)]\,\frac{\mathrm{d}\sigma}{2\pi}+\int_0^{2\pi} f^2(\sigma)\,\frac{\mathrm{d}\sigma}{2\pi}\right)$$

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$$\geq \ell^2 + \left(\ell \int_0^{2\pi} \cot[\tau(\sigma)] \frac{\mathrm{d}\sigma}{2\pi}\right)^2$$

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$$= \ell^2 + \left(\frac{L}{2\pi}\right)^2 \qquad \square$$

Thank you for listening!