AdS nonlinear instability:

moving beyond spherical symmetry

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Based on:

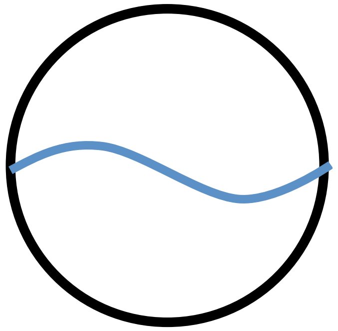
OD, Jorge Santos, 1602.03890

OD, Gary Horowitz, Jorge Santos, 1109.1825

OD, Gary Horowitz, Don Marolf, Jorge Santos, 1208.5772

GR21, D2 session: Strings, branes, entanglement, AdS/CFT, and all that **New York, July 2016**

→ Non-linear Instability of AdS: the origins



- Linear perturbations in AdS do <u>not</u> decay: normal modes $\omega L = 1 + \ell + 2p$

  => conjecture (Dafermos-Holzegel, 2006):

  it should be non-linearly unstable

- **Consistent with**

  time evolution of spherical scalar field shell in AdS: collapse to BH

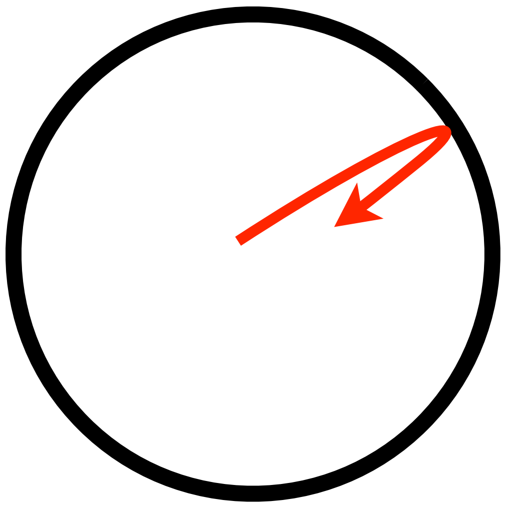
[Bizon-Rostworowski 2011]

No matter how small

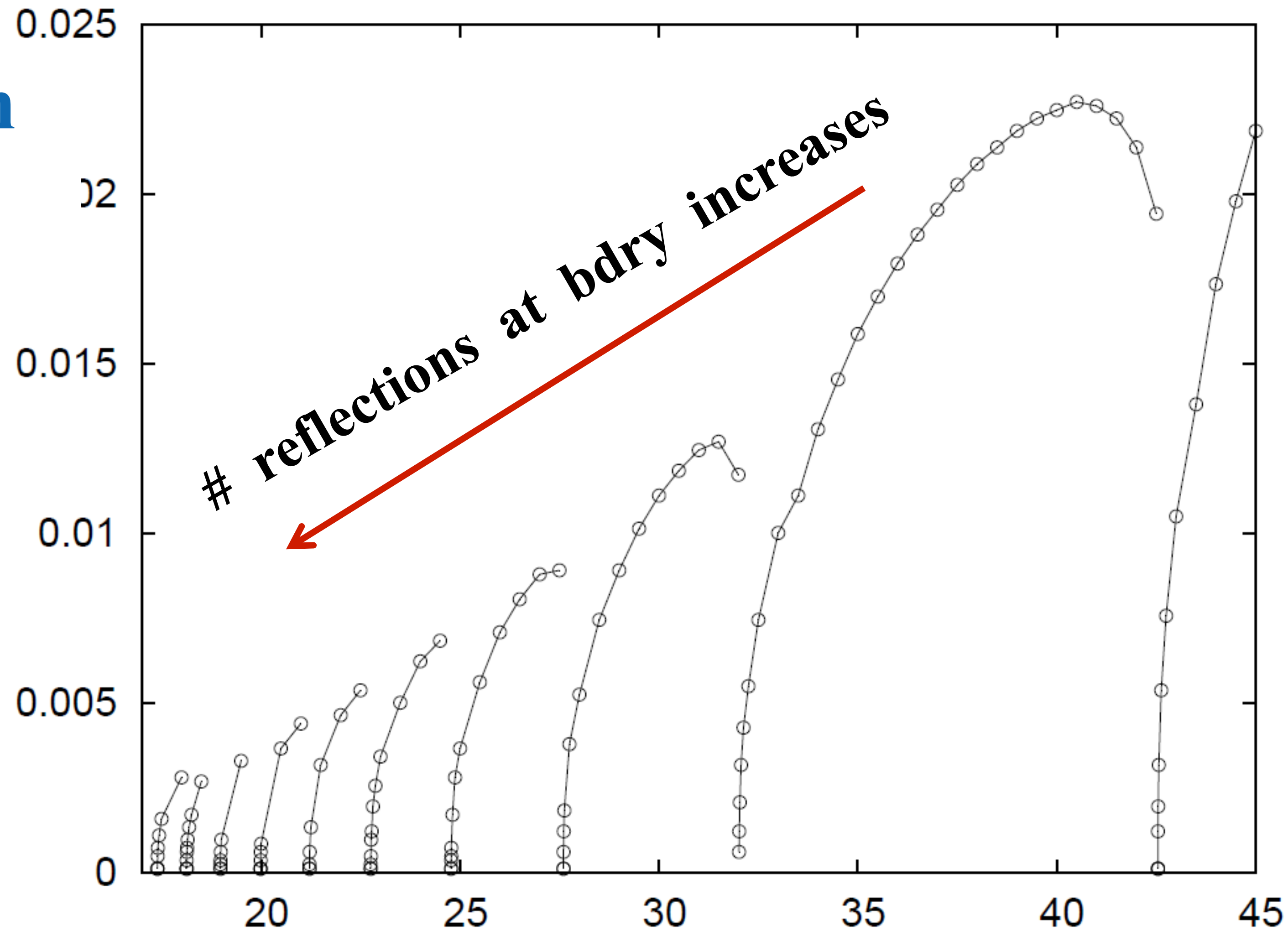
the initial amplitude ε is,

the curvature at the origin

grows & a small BH forms.



Horizon radius



Amplitude of initial perturbation

→ Understanding the onset of the instability using perturbation theory:

- **Weakly perturbative turbulent instability:**
  - A **secular term** of the form $\varepsilon^3 t$ appears at 3rd order in the **amplitude** ε of linear seed
  - **Necessary** condition for secular turbulent growth: **linear spectrum is fully commensurable**

[OD, Horowitz, Marolf, Santos, 2012]

- **Improved perturbation theory that captures the dynamics up to time scales** $t \lesssim \varepsilon^{-2}$
  - two time scale formalism [Balasubramanian, Buchel, Green, Lehner, Liebling, 2014]
  - renormalisation group perturbation methods [Craps, Evnin, Vanhoof, 2014]
  - resonant approx. [Bizon, Maliborski, Rostworowski, 2015]

- **Question: Is this a fine-tunning process** (spherical symmetry) ?

Consider non-spherically symmetric gravitational modes.

Includes rotating modes: can centrifugal effects balance grav. collapse ?

→ Weakly perturbative turbulent analysis shows that this is NOT the case.

[OD, Horowitz, Santos, 2011]

→ Understanding the onset of the instability using perturbation theory:

• Is this a fine-tunning process (spherical symmetry) ?

Consider non-spherically symmetric gravitational modes:

Includes rotating modes: can centrifugal effects balance grav. collapse ?

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[OD, Horowitz, Santos, 2011]

... but this is not the whole story... interesting twist:

non-spherical grav. modes favour the AdS non-linear instability

➙ Technical approach to study grav. sector: (standard) perturbation theory

- Expand the metric around global AdS as $$g = \bar{g} + \sum_i \epsilon^i h^{(i)}$$

  At each order (i) in perturbation theory, Einstein's equations yield:

  $$\tilde{\Delta}_L h^{(i)}_{ab} = T^{(i)}_{ab}$$

  where $T^{(i)}$ depends on $\{ h^{(j \leq i-1)} \}$ and their derivatives and

  $$2\tilde{\Delta}_L h^{(i)}_{ab} \equiv -\bar{\nabla}^2 h^{(i)}_{ab} - 2\bar{R}_a{}^c{}_b{}^d h^{(i)}_{cd} - \bar{\nabla}_a \bar{\nabla}_b h^{(i)} + 2\bar{\nabla}_{(a}\bar{\nabla}^c h^{(i)}_{b)c}$$

- Expand in terms of spherical harmonics: $h_{ab} \sim e^{i m \phi} Y_{\ell m}(\theta)$

ℓ counts # of nodes

- There are two sectors: scalar and vector harmonics

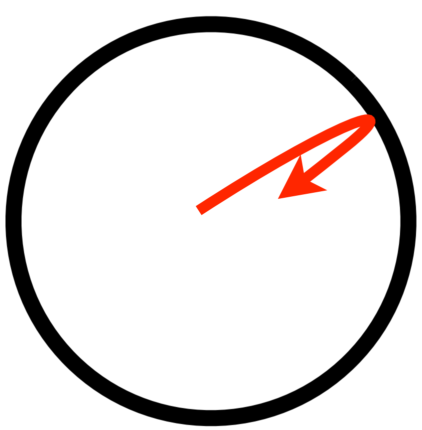
→ Boundary conditions

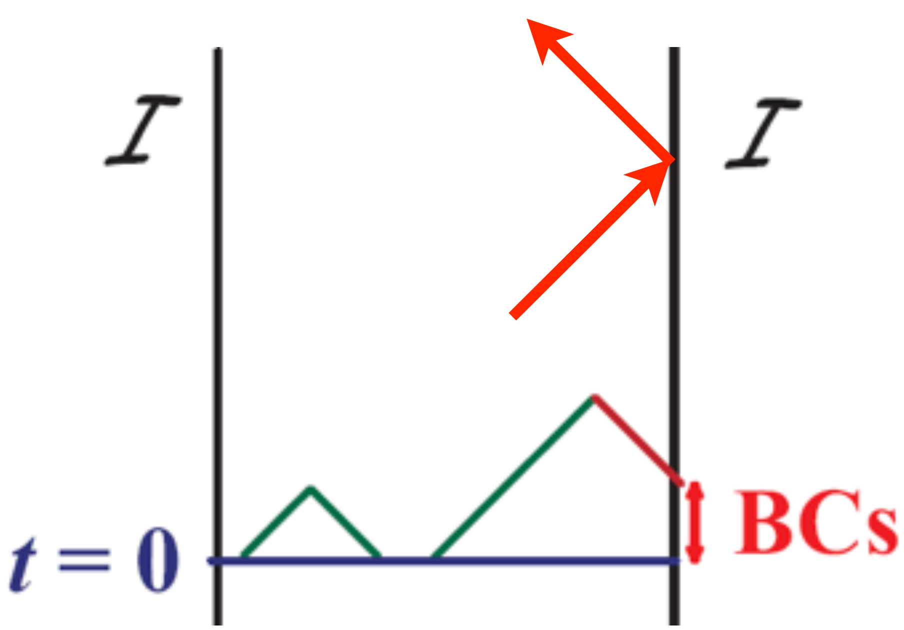
- **Regularity at the origin**

- **Keep boundary metric fixed**

  (ie perturbations preserve global AdS asymptotics)

These BCs also **preserve** ***E, J***





→ General Structure of perturbation problem

- **Non-linearity of Einstein's eqs:**

  start with *a* given pair $\{\ell\,,\,m\}$ → it generates several $\{\ell\,,\,m\}$'s.

  1^{st} task: identify decomposition of T as a sum of $\{\ell\,,\,m\}$'s :

$$T^{(i+1)}(h^{j\leq i}) = \sum_{\ell,m} T^{(i+1)}_{\ell m}$$

- **If** $T^{(i+1)}_{\ell m}(t,r)$ **has an harmonic time dependence $\cos(\omega\, t)$, then**

  $h^{(i+1)}_{\ell m}(t,r)$ **will exhibit** the **same dependence ,**

  EXCEPT when ω agrees with one of the **normal frequencies of AdS:**

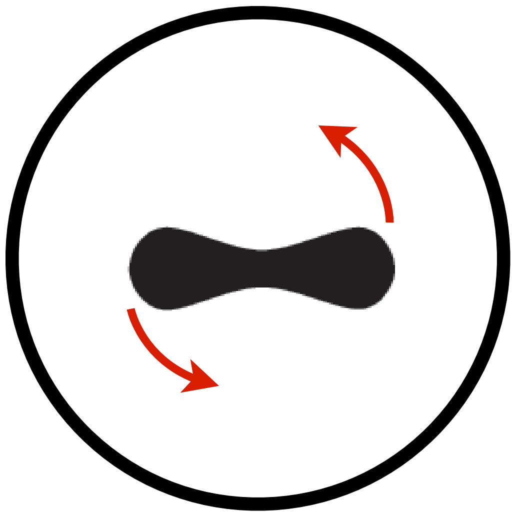
$$h^{(i+1)}_{\ell m} = H_c(r)\cos(\omega t) + H_s(r)\; \boldsymbol{t}\; \sin(\omega t)$$

$$\boxed{\textbf{Mode is said to be } \textbf{\textit{RESONANT}}}$$

- **Some resonances can be removed** with a **frequency correction** …

  If not, AdS is non-linearly unstable

➙ Back-reaction of SINGLE grav. normal mode can already trigger secular resonances



- ***ONLY*** **grav. normal modes** that can be **back-reacted up** to $\boldsymbol{O(\varepsilon^3)}$ to yield a **time-periodic soliton (GEON)** are:
  - — **Scalar modes** with $\ell = m \geq 2$ and $p=0$,
  - — **Scalar modes** with $\ell = 2$, $m=0,1$ and $p=0$,
  - — **Vector modes** with $\ell = 2$, $m=0$ and $p=0$.

- $\{\ell, m, p\}=\{2,2,0\}$ case was back-reacted up to $O(\varepsilon^5)$
  … actually up to ***any*** **order**: **full nonlinear** extension to a geon **[Horowitz, Santos, 2014]**.

- **'Few' normal modes** with a **solitonic extension:** ***unique*** to the **gravitational sector**.
  Real (complex) scalar field: back-reaction of ***any*** normal mode yields an **oscillon (boson star)**.

- **Due to gravity** having ***two*** **fundamentally distinct sectors** (*scalar* and *vector*) of normal modes whose ω spectra depend on **two quantum #** (not one): ℓ and p.

So **different** $\{\ell, p\}$**'s** can **yield** the **same** ω ➙ system **more prone** to develop **secular resonances**

Spherically symmetric scalar field has a ***single*** **normal mode** sector; **spectrum** depends ***only*** on p).

➙ **Consequences when we have a seed with $n \geq 2$ normal modes:**

- **For spherically symmetry scalar field collapse:**

  Collision of **two normal modes** ***always*** yields ***only*** a ***pair*** of **irremovable resonances.**

- **Collision of two gravitational normal modes** that do ***not*** have a **geon extension** generates ***more than just a pair*** of **irremovable resonances**

- ***Strong evidence suggesting*** that the **time evolution** of the **gravitational nonlinear** instability should be **more 'dramatic' and possibly 'even faster'** than **spherical symmetric scalar field.**

  (although our pert. theory analysis still breaks down at $O(\varepsilon^3)$ and thus for timescales above $O(\varepsilon^{-2})$)

- Moduli space of **rotating black hole solutions in AdS** **[see Way's talk]:**

  — $r_+ \to 0$ limit of a (superradiant) `**black resonator**' is a geon only for $\ell = m \geq 2, p = 0$.

  — Otherwise, limit likely to be **singular**

➞ Direct and inverse turbulent cascades

- Seed with a ***single* normal mode**: ω of **secular resonances** = ω **of seed**
- **Smoking gun** of a ω **cascade**: secular resonances with ω different from seed.

=> starting with **superposition** (collision) of at least ***two*** normal **modes**.

➞ For example, take the seed:

$$\{\ell, m, p, \bar{\omega}\}_{\mathbf{s}} = \{4, 4, 0, 5/L\}, \quad \text{amplitude } \mathcal{A}^{(1)}_{(\mathbf{s})4}\varepsilon,$$

$$\{\ell, m, p, \bar{\omega}\}_{\mathbf{s}} = \{6, 6, 0, 7/L\}, \quad \text{amplitude } \mathcal{A}^{(1)}_{(\mathbf{s})6}\varepsilon,$$

- Two resonances, $\{\ell, m, p, \omega\}_s = \{4,4,0, \mathbf{5/L}\}$ and $\{\ell, m, p, \omega\}_s = \{6,6,0, \mathbf{7/L}\}$, removed with **Poincaré-Lindstedt ω correction:**

$$\omega = \bar{\omega} + \varepsilon^2\omega^{(2)} + \mathcal{O}(\varepsilon^4),$$ for a choice of $\omega_{4,4,0}^{(2)}$ & $\omega_{6,6,0}^{(2)}$

- But, also two ***irremovable* secular resonances**, $\{\ell, m, p, \omega\}_s = \{\{2,2,0, \mathbf{3/L}\},\{8,8,0, \mathbf{9/L}\}\}$ whose **quantum # do *not* coincide with seed:**

  — First mode: $\boldsymbol{\omega = 3/L}$ ➞ ***smaller*** than two **ω's** of **seed**.

  — Second mode: $\boldsymbol{\omega = 9/L}$ ➞ ***larger*** than two **ω's** of **seed**.

➞ So, **weakly pert. turbulent mechanism** predicts **generation** of **irremovable resonances** with ***both*** **larger** and **smaller ω** **than those of the seed**

=> **time evolution** should proceed with ***direct*** and ***inverse*** **ω cascades**.

→ **which of the cascades is likely to dominate faster the time-evolution?**

Compare the coefficient of the direct and inverse cascades (in a **gauge invariant way**):

1) Compute boundary **holographic stress energy tensor** and associated **energy density**.

2) **compare** the **ratio between** the **two secular terms**.

If we **assume** that **each of the modes** in the seed carries ***equal energy***,

the **direct cascade is a factor of 10 larger** than the inverse cascade,

perhaps signalling that **black hole formation is likely to occur at late times**.

→ Conclusions:

- Surprisingly, centrifugal effects cannot balance grav. collapse

  i.e. cannot halt AdS non-linear instability

- Actually, non-spherical grav. modes favour the AdS non-linear instability

- Only a few gravitational normal modes (scalar modes with $\ell = m \geq 2$, p=0)

  can be back-reacted to non-linear order to yield a geon

- Weakly perturbative turbulent analysis:
  - predicts the existence of both direct and inverse frequency cascades
  - suggests the former should dominate the late time evolution

Normal mode $\{\ell,m,p,\bar{\omega}\}$ at $\mathcal{O}(\varepsilon)$	# modes $\mathcal{O}(\varepsilon^2)$	# modes $\mathcal{O}(\varepsilon^3)$	Removable resonance $\left(-L\,\omega^{(2)}\right)$	Secular resonances $\{\ell,m,p,\omega\}$
$\{\mathbf{2},\mathbf{0},\mathbf{0},\frac{\mathbf{3}}{\mathbf{L}}\}_{\mathbf{s}}$	$6_{\mathbf{s}}$ $0_{\mathbf{v}}$	$8_{\mathbf{s}}$ $0_{\mathbf{v}}$	$\{2,0,0,\frac{3}{L}\}_{\mathbf{s}}$ $\left(\frac{3663}{8960}\right)$	None **(Geon ?)**
$\{2,0,1,\frac{5}{L}\}_{\mathbf{s}}$	$6_{\mathbf{s}}$ $0_{\mathbf{v}}$	$8_{\mathbf{s}}$ $0_{\mathbf{v}}$	$\{2,0,1,\frac{5}{L}\}_{\mathbf{s}}$ $\left(\frac{34397}{5376}\right)$	$\{4,0,0,\frac{5}{L}\}_{\mathbf{s}}$
$\{4,0,0,\frac{5}{L}\}_{\mathbf{s}}$	$10_{\mathbf{s}}$ $0_{\mathbf{v}}$	$14_{\mathbf{s}}$ $0_{\mathbf{v}}$	$\{4,0,0,\frac{5}{L}\}_{\mathbf{s}}$ $\left(\frac{52311625}{21446656}\right)$	$\{2,0,1,\frac{5}{L}\}_{\mathbf{s}}$
$\{\mathbf{2},\mathbf{1},\mathbf{0},\frac{\mathbf{3}}{\mathbf{L}}\}_{\mathbf{s}}$	$5_{\mathbf{s}}$ $2_{\mathbf{v}}$	$5_{\mathbf{s}}$ $4_{\mathbf{v}}$	$\{2,1,0,\frac{3}{L}\}_{\mathbf{s}}$ $\left(\frac{123}{64}\right)$	None **(Geon ?)**
$\{\mathbf{2},\mathbf{2},\mathbf{0},\frac{\mathbf{3}}{\mathbf{L}}\}_{\mathbf{s}}$	$4_{\mathbf{s}}$ $2_{\mathbf{v}}$	$4_{\mathbf{s}}$ $2_{\mathbf{v}}$	$\{2,2,0,\frac{3}{L}\}_{\mathbf{s}}$ $\left(\frac{14703}{1120}\right)$	None **(Geon)**
$\{2,2,1,\frac{5}{L}\}_{\mathbf{s}}$	$4_{\mathbf{s}}$ $2_{\mathbf{v}}$	$4_{\mathbf{s}}$ $2_{\mathbf{v}}$	$\{2,2,1,\frac{5}{L}\}_{\mathbf{s}}$ $\left(\frac{9409723}{70560}\right)$	$\{4,2,0,\frac{5}{L}\}_{\mathbf{s}}$ $\{3,2,0,\frac{5}{L}\}_{\mathbf{v}}$
$\{\mathbf{3},\mathbf{3},\mathbf{0},\frac{\mathbf{4}}{\mathbf{L}}\}_{\mathbf{s}}$	$5_{\mathbf{s}}$ $3_{\mathbf{v}}$	$5_{\mathbf{s}}$ $3_{\mathbf{v}}$	$\{3,3,0,\frac{4}{L}\}_{\mathbf{s}}$ $\left(\frac{27881625}{32032}\right)$	None **(Geon)**
$\{3,2,0,\frac{4}{L}\}_{\mathbf{s}}$	$5_{\mathbf{s}}$ $2_{\mathbf{v}}$	$6_{\mathbf{s}}$ $5_{\mathbf{v}}$	$\{3,2,0,\frac{4}{L}\}_{\mathbf{s}}$ $\left(\frac{8081875}{72072}\right)$	$\{2,2,0,\frac{4}{L}\}_{\mathbf{v}}$
$\{\mathbf{4},\mathbf{4},\mathbf{0},\frac{\mathbf{5}}{\mathbf{L}}\}_{\mathbf{s}}$	$6_{\mathbf{s}}$ $4_{\mathbf{v}}$	$6_{\mathbf{s}}$ $4_{\mathbf{v}}$	$\{4,4,0,\frac{5}{L}\}_{\mathbf{s}}$ $\left(\frac{7010569125}{77792}\right)$	None **(Geon)**
$\{4,2,0,\frac{5}{L}\}_{\mathbf{s}}$	$8_{\mathbf{s}}$ $4_{\mathbf{v}}$	$10_{\mathbf{s}}$ $7_{\mathbf{v}}$	$\{4,2,0,\frac{5}{L}\}_{\mathbf{s}}$ $\left(\frac{163492329375}{243955712}\right)$	$\{2,2,1,\frac{5}{L}\}_{\mathbf{s}}$ $\{3,2,0,\frac{5}{L}\}_{\mathbf{v}}$
$\{\mathbf{6},\mathbf{6},\mathbf{0},\frac{\mathbf{7}}{\mathbf{L}}\}_{\mathbf{s}}$	$8_{\mathbf{s}}$ $6_{\mathbf{v}}$	$8_{\mathbf{s}}$ $6_{\mathbf{v}}$	$\{6,6,0,\frac{7}{L}\}_{\mathbf{s}}$ $\left(-L\,\omega^{(2)}_{6,6,0}\right)$	None **(Geon)**

Normal mode $\{\ell,m,p,\bar{\omega}\}$ at $\mathcal{O}(\varepsilon)$	# modes $\mathcal{O}(\varepsilon^2)$	# modes $\mathcal{O}(\varepsilon^3)$	Removable resonance $\left(-L\,\omega^{(2)}\right)$	Secular resonances $\{\ell,m,p,\omega\}$
$\{\mathbf{2},\mathbf{0},\mathbf{0},\frac{\mathbf{4}}{\mathbf{L}}\}_{\mathbf{v}}$	$6_{\mathbf{s}}$ $0_{\mathbf{v}}$	$0_{\mathbf{s}}$ $6_{\mathbf{v}}$	$\{2,0,0,\frac{4}{L}\}_{\mathbf{v}}$ $\left(\frac{1469}{26880}\right)$	None **(Geon ?)**
$\{2,0,1,\frac{6}{L}\}_{\mathbf{v}}$	$6_{\mathbf{s}}$ $0_{\mathbf{v}}$	$0_{\mathbf{s}}$ $6_{\mathbf{v}}$	$\{2,0,1,\frac{6}{L}\}_{\mathbf{v}}$ $\left(\frac{19081}{376320}\right)$	$\{4,0,0,\frac{6}{L}\}_{\mathbf{v}}$
$\{2,1,0,\frac{4}{L}\}_{\mathbf{v}}$	$5_{\mathbf{s}}$ $2_{\mathbf{v}}$	$4_{\mathbf{s}}$ $5_{\mathbf{v}}$	$\{2,1,0,\frac{4}{L}\}_{\mathbf{v}}$ $\left(\frac{72361}{322560}\right)$	$\{3,1,0,\frac{4}{L}\}_{\mathbf{s}}$
$\{2,2,0,\frac{4}{L}\}_{\mathbf{v}}$	$4_{\mathbf{s}}$ $2_{\mathbf{v}}$	$2_{\mathbf{s}}$ $4_{\mathbf{v}}$	$\{2,2,0,\frac{4}{L}\}_{\mathbf{v}}$ $\left(\frac{1247}{1008}\right)$	$\{3,2,0,\frac{4}{L}\}_{\mathbf{s}}$
$\{3,2,0,\frac{5}{L}\}_{\mathbf{v}}$	$5_{\mathbf{s}}$ $3_{\mathbf{v}}$	$5_{\mathbf{s}}$ $6_{\mathbf{v}}$	$\{3,2,0,\frac{5}{L}\}_{\mathbf{v}}$ $\left(\frac{31995875}{4612608}\right)$	$\{2,2,1,\frac{5}{L}\}_{\mathbf{s}}$ $\{4,2,0,\frac{5}{L}\}_{\mathbf{s}}$
$\{7,6,0,\frac{9}{L}\}_{\mathbf{v}}$	$10_{\mathbf{s}}$ $7_{\mathbf{v}}$	$9_{\mathbf{s}}$ $8_{\mathbf{v}}$	$\{7,6,0,\frac{9}{L}\}_{\mathbf{v}}$ $\left(-L\,\omega^{(2)}_{7,6,0}\right)$	$\{6,6,1,\frac{9}{L}\}_{\mathbf{s}}$ $\{8,6,0,\frac{9}{L}\}_{\mathbf{s}}$

Some properties of geons:

- Geon: regular horizonless solutions of Einstein-AdS
- Invariant under single helical Killing vector field: $K = \partial_t + \frac{\omega}{m} \partial_\varphi$

  which is timelike near the poles but spacelike near the equator.

  Thus, it is not time symmetric neither axisymmetric but time-periodic
- Obeys the first law: $dE = (\omega/m)\, dJ$
- From the QFT perspective do not seem to thermalize.

  Boundary stress-tensor has regions of negative and positive energy density around the equator: