Conformal properties of the Schwarzschild-de Sitter spacetime

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based on arXiv:1506.00030v2
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CONACYT

The Einstein equations under conformal transformations

Recall that:

- The Einstein field equations

$$\tilde{R}_{ab} = \lambda \tilde{g}_{ab}$$

  are not conformally invariant!.

- A calculation shows that if

$$g_{ab} = \Omega^2 \tilde{g}_{ab},$$

  then $\qquad R_{ab} - \frac{1}{2} R g_{ab} = -\frac{2}{\Omega} \nabla_a \nabla_b \Omega$

  $- \left(\frac{1}{\Omega} \nabla_c \nabla^c \Omega - \frac{3}{\Omega^2} \nabla_c \Omega \nabla^c \Omega - \frac{1}{\Omega^2} \lambda \right) g_{ab},$

  where R_{ab}, R and ∇_a are associated to g_{ab}.

- This equation is not a good equation for the components of the metric g_{ab} as it is formally **singular** whenever $\Omega = 0$.

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A regular set of equations [Friedrich, 1981]

Introducing the curvature as an unknown and reading the singular conformal Einstein field equation as an equation for the conformal factor one can obtain a set of regular **conformal Einstein field equations** (CEFE).

$$\nabla_a\nabla_b\Omega = -\Omega L_{ab} + sg_{ab},$$

$$\nabla_a s = -L_{ac}\nabla^c\Omega,$$

$$\nabla_c L_{db} - \nabla_d L_{cb} = \nabla_a\Omega d^a{}_{bcd},$$
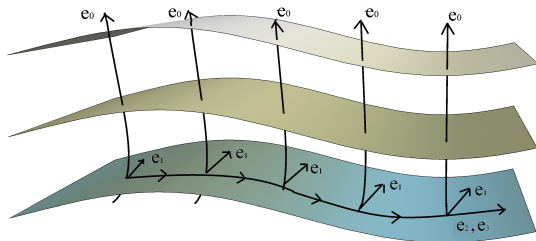
$$\nabla_a d^a{}_{bcd} = 0,$$

$$6\Omega s - 3\nabla_c\Omega\nabla^c\Omega = \lambda,$$

$$\{L_{ab}, s, \Omega, d^a{}_{bcd} \equiv \Omega^{-1}C^a{}_{bcd}\}$$

The conformal Einstein field equations

- Global problems in $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ recast as local problems in $(\mathcal{M}, \boldsymbol{g})$.
- SCEFE : Standard $\rightarrow$ Hyperbolic reduction via gauge source functions
- XCEFE: Extended $\rightarrow$ Hyperbolic reduction using a congruence of **conformal geodesics**
- Conformal geodesics $\rightarrow$ locus of $\mathscr{I}$ known apriori
- Solution to the CEFE $\rightarrow$ EFE

$$\tilde{\boldsymbol{g}} = \Omega^{-2}\boldsymbol{g}$$

- The XCEFE expressed in terms of a **conformal Gaussian system** imply an evolution system of the form:

$$\partial_\tau \hat{\boldsymbol{v}} = \mathbf{K}\hat{\boldsymbol{v}} + \mathbf{Q}(\hat{\Gamma})\hat{\boldsymbol{v}} + \mathbf{P}(x)\boldsymbol{\phi},$$

$$(\mathbf{I} + \mathbf{A}^0(\boldsymbol{e}))\partial_\tau \boldsymbol{\phi} + \mathbf{A}^\alpha(\boldsymbol{e})\partial_\alpha \boldsymbol{\phi} = \mathbf{B}(\hat{\Gamma})\boldsymbol{\phi}$$

$$\hat{\boldsymbol{v}} \quad \leftrightarrow \quad \boldsymbol{e_a}, \quad \hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}, \quad \hat{L}_{\boldsymbol{ab}},$$

$$\boldsymbol{\phi} \quad \leftrightarrow \quad d^{\boldsymbol{a}}{}_{\boldsymbol{bcd}}$$

The Schwarzschild-de Sitter spacetime

The **Schwarzschild-de Sitter spacetime** in standard coordinates (t, r, θ, φ) is determined by the line element
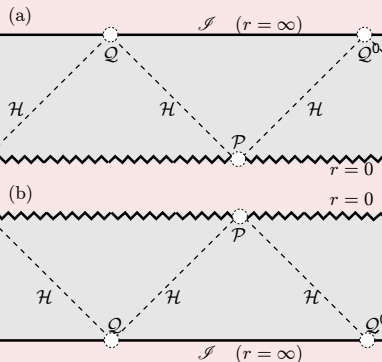
$$\tilde{\boldsymbol{g}} = -\left(1 - \frac{2m}{r} - \frac{\lambda r^2}{3}\right) \mathbf{d}t \otimes \mathbf{d}t + \left(1 - \frac{2m}{r} - \frac{\lambda r^2}{3}\right)^{-1} \mathbf{d}r \otimes \mathbf{d}r + r^2 \boldsymbol{\sigma}.$$
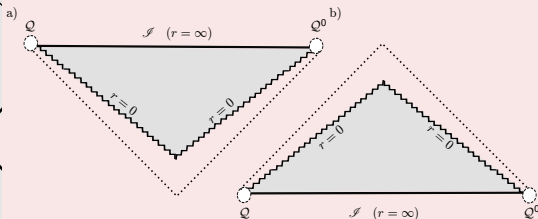
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Subextremal

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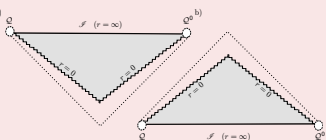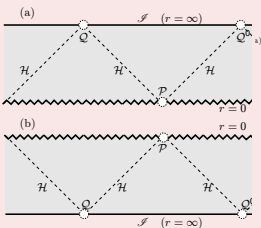

Extremal
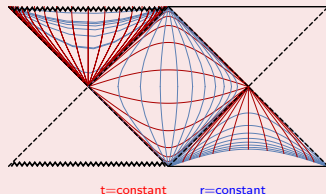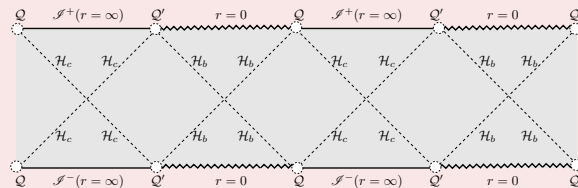
Hyperextremal

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Conformal geodesics in SdS and initial data

No explicit expression for the SdS metric in the required Gauge!

Although the **Schwarzschild-de Sitter** spacetime is an explicitly known solution, it is very hard to recast explicitly it in terms of a conformal Gaussian gauge hinged at $\mathscr{I}$

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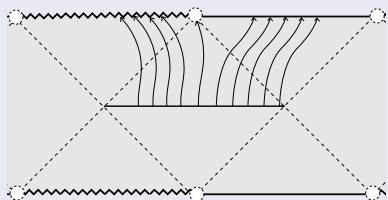
Asymptotic initial data

All the fields in the initial data except for the rescaled Weyl tensor, are either gauge quantities or can be deduced from the intrinsic metric at $\mathscr{I}$

A conformal Gaussian system

- The past domain of dependence of $\mathscr{I}$ can be covered by a non-intersecting congruence of conformal geodesics starting **orthogonal to the conformal boundary**. **[EG., A Garcia-Parrado, Valiente Kroon]**

Induced metric at $\mathscr{I}$

- The intrinsic metric of $\mathscr{I}$ is conformally flat.

- There exists a conformal representation of $\mathscr{I}$ in which the asymptotic points $\mathcal{Q}$ and $\mathcal{Q}'$ correspond to the North and South poles of $\mathbb{S}^3$.

- No timelike conformal geodesics reach the asymptotic points $\mathcal{Q}$ and $\mathcal{Q}'$.

The rescaled Weyl tensor at $\mathscr{I}$

- The initial data for d_{ij} satisfies the **Gauss constraint**

$$D^i d_{ij} = 0.$$

This equation well understood in the conformally flat setting **[Dain & Friedrich (2001)]**.

- Spherical symmetry requires $\mathcal{L}_\xi d_{ij} = 0$ for each of the Killing vectors generating the spherical symmetry **[Paetz (2014)]**

$$d_{ij} = \frac{m}{r^3}\left(3\frac{x_i x_j}{r^2} - \delta_{ij}\right).$$

This solution is singular at the origin.

- Using the stereographic projection this solution can be transformed into a solution on $\mathbb{S}^3$ which is singular at the poles $\rightarrow \mathcal{Q}$ and $\mathcal{Q}'$.

The $\mathbb{S}^3$ representation

- $\mathscr{I}$ as a 3-manifold is smooth. However, as a submanifold of the 4-dimensional unphysical spacetime it contains singular points: $\mathcal{Q}$ and $\mathcal{Q}'$.

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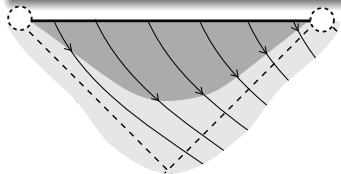
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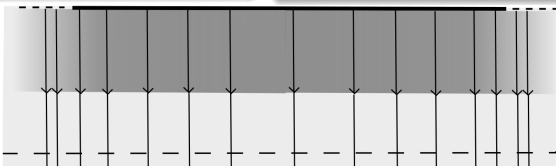
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The $\mathbb{R} \times \mathbb{S}^2$ representation

- $\mathbb{S}^3$ conformal to $\mathbb{R} \times \mathbb{S}^2$: send North and South poles to ∞.

- In the $\mathbb{R} \times \mathbb{S}^2$ rep. d_{ij} is regular and homogenous

The spherically symmetric conformal evolution eqns

The core system:

- The dynamics of the system is governed by a **core system** which decouples from the rest of the equations:

$$\dot{\phi} = -3\chi\phi,$$

$$\dot{\chi} = -\chi^2 + L - \frac{1}{2}\Theta\phi,$$

$$\dot{L} = -\chi L - \frac{1}{2}\dot{\Theta}\phi$$

where

$$\Theta = \sqrt{\frac{\lambda}{3}}\tau\left(1 + \frac{1}{2}\varkappa\tau\right), \qquad \dot{\Theta} = \sqrt{\frac{\lambda}{3}}(1 + \varkappa\tau).$$
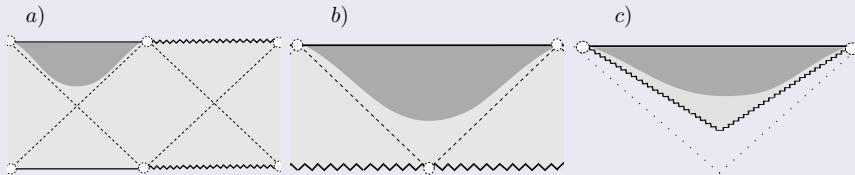
- The initial data is given by

$$\phi(0) = 2m, \qquad \chi(0) = \varkappa, \qquad L(0) = \frac{1}{2}(1 - \varkappa^2).$$

Theorem (E.G., J.A. Valiente Kroon, 2015)

Given asymptotic initial data which is suitably close to data for the Schwarzschild-de Sitter spacetime, in a suitable conformal representation, there exists a solution to Einstein field equations which exists towards the future (past) and has an asymptotic structure similar to that of the Schwarzschild-de Sitter spacetime —that is, the solution is future (past) weakly asymptotically simple.

$\mathbb{S}^3$- representation

$\mathbb{R} \times \mathbb{S}^2$- representation

Questions?

Thanks for your attention

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