Quantum Resolution of Time-like Classical Singularities

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SINGULARITIES

"We…have dreamt the world. We have dreamt it as firm, mysterious, visible, ubiquitous in space and durable in time; but in its architecture we have allowed tenuous and eternal crevices of unreason which tell us it is false."

- Jorge Luis Borges

Outline

1. Singularities
    a. Classical
    b. Quantum

2. Brief History

3. Conformally Static Spacetimes

4. Discussion

Space-time (M,g)

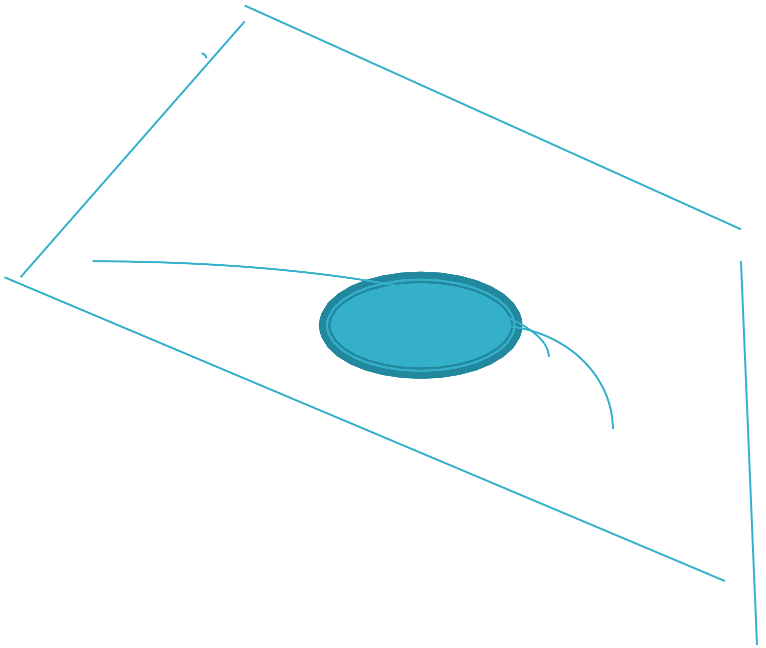
- smooth, C^{∞}, paracompact, connected Hausdorff manifold M
- Lorentzian metric g

Singularity????

- Space–time is smooth!

- "singular" point must be cut out of space–time

$\Rightarrow$ leaves hole
$\Rightarrow$ incomplete curves
$\Rightarrow$ boundary points



How do we define a boundary ∂M to space-time????

Complete: ST (M, g) + Boundary (∂M)

Cauchy completeness only works with Riemannian metric, not Lorentzian.

How do we complete ST?

Singularities as Boundary Points

a (abstract) – boundary
(Scott and Szekeres (1994))

b (bundle) – boundary
(Schmidt (1971))

c (causal) – boundary
(Geroch, Kronheimer and Penrose (1972))

g (geodesic) – boundary
(Geroch (1968))

Classical Singularity

" a singularity is indicated by incomplete geodesics or incomplete curves of bounded acceleration in a maximal space–time" (Geroch (1968))

Quantum Waves
vs.
Classical Geodesics

What happens if instead of classical particle paths (timelike and null geodesics) one used quantum mechanical particles (QM waves) to identify singularities????

Horowitz and Marolf (1995):

A space-time is QM nonsingular if the evolution of a test scalar wave packet, representing a quantum particle, is uniquely determined by the initial wave packet, manifold and metric, without having to put boundary conditions at a classical singularity.

TECHNICALLY: A static ST is QM-singular if the spatial portion of the Klein-Gordon operator is not essentially self-adjoint on $C_0^\infty(\Sigma)$ in $L^2(\Sigma)$.

Essentially Self- Adjoint

An operator, A, is called self-adjoint if

(i) $A = A^*$

(ii) $\mathrm{Dom}(A) = \mathrm{Dom}(A^*)$

where A^* is the adjoint of A.

An operator is essentially self-adjoint if (i) is met and (ii) can be met by expanding the domain of the operator or its adjoint so that it is true.

Tests for Essential Self-Adjointness

1. von Neumann criterion of deficiency indices: study solutions to $A^*\Psi = \pm i\Psi$, where A is the spatial K-G operator and find the number that are self-adjoint for each i.

2. Weyl limit point-limit circle criterion: relate essential self-adjointness of Hamiltonian operator to behavior of the "potential" in an effective '1D Schrodinger Eq.' (made from the K-G radial eq.), which in turn determines the behavior of the scalar wave packet.

Weyl
limit point – limit circle criterion

Rewrite K–G radial equation in
"1D Schrodinger form" $H\Psi(x) = E\Psi(x)$
where the operator
$H = -d^2/dx^2 + V(x)$ and E is a constant
and
any singularity is taken to be at $x = 0$.

Weyl LP – LC Criterion Cont.

Reed and Simon (1972):

Definition: The potential $V(x)$ is in the limit circle case at $x = 0$ (∞) if for some, and therefore all E, the solutions of $H\Psi = E\Psi$ are square integrable at zero (∞). If $V(x)$ is not in the limit circle case, it is in the limit point case.

If LC we would need a BC at $x = 0$ to establish unique solution; if LP we do not need one. Whole point behind QM singularity.

Weyl's Theorem (1910)

If $V(x)$ is a continuous real-valued function on $(0,\infty)$, then $H = -d^2/dx^2 + V(x)$ is essentially self-adjoint on $C_o^\infty(0,\infty)$ if and only if $V(x)$ is in the limit point case at both zero and infinity.

A useful theorem near infinity (see RS(1972)) can be used – it establishes LP for all the STs considered here. We won't worry about infinity in what follows.

Useful Theorem (X.10 of RS (1972))

Let V(x) be continuous and positive near zero.
If $V(x) \geq ¾\, x^{-2}$ near zero,
then V(x) is in the limit point case.
If for some $\epsilon > 0$,
$V(x) \leq (3/4 - \epsilon)\, x^{-2}$ near zero,
then V(x) is in the limit circle case.

Thus potential is only LP if it is sufficiently repulsive near the origin that one of the two solutions of the '1D Schrodinger Eq'. blows up so quickly it fails to be square-integrable.

Technique

To study the quantum particle propagation in spacetimes, we use scalar particles described by the Klein-Gordon equation and the limit circle-limit point criterion of Weyl (1910).

In particular, we study the radial equation in a one-dimensional Schrodinger form with a ‘potential’ and determine the number of solutions that are square integrable. If we obtain a unique solution, without placing boundary conditions at the location of the classical singularity, we can say that the spacetime is quantum mechanically nonsingular.

The results depend on spacetime metric parameters and wave equation modes.

Brief History

Basic References:

1. R.M. Wald, "Dynamics in non-globally hyperbolic, static spacetimes," *J. Math. Phys.* **21**, 2802 (1980)

2. G.T. Horowitz and D. Marolf, "Quantum probes of spacetime singularities," *Phys. Rev. D* **52**, 5670 (1995)

3. A. Ishibashi and A. Hosoya, " Who's afraid of naked singularities? Probing timelike singularities with finite energy waves," *Phys. Rev. D* **60**, 104028 (1999)

Review Articles on Quantum Singularities

- Quantum Singularities in Static Spacetimes.
- Joao Paulo M. Pitelli, Patricio S. Letelier (Campinas State U., IMECC). Oct 2010. 14 pp.
- Published in **Int.J.Mod.Phys. D20 (2011) 729-743**
- e-Print: **arXiv:1010.3052 [gr-qc]**

Includes:

- Math Review
- Cosmic String
- Global Monopole
- BTZ Space-time (massive field – QM singular;
  massless field – QM non-singular)

cont.

- Quantum singularities in static and conformally static space-times
- D.A. Konkowski, T.M. Helliwell
- Comments: 16 pages, 8th Friedmann Seminar, Rio de Janeiro, Brazil, 30 May - 3 June, 2011
- Journal-ref: International Journal of Modern Physics A, Vol.26, No.22 (2011) 3878-3888

  arXiv:1112.5488 (gr-qc)

Conformally Static Space-Times

Test for Quantum Singularity
using
Conformally Static ST
and
Generally Coupled Scalar Field
and
Associated Inner Product

Idea comes from Ishibashi and Hosoya (1999) who suggested a conformally coupled scalar field and associated inner product for "wave regularity" (Sobolev norm).

Klein-Gordon w/ general coupling

is

$$|g|^{-1/2} \left(|g|^{1/2} g^{\alpha\beta} \Phi_{,\beta}\right)_{,\alpha} = \xi R \Phi$$

where

ξ is the coupling constant
and
R is the scalar curvature

Conformally Static Metric and Inner Product

If the conformally static metric has the form

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = C^2(t)(\gamma_{tt} dt^2 + \gamma_{ab} dx^a dx^b)$$

where a,b=1,2,3, if χ and ζ are mode solutions of the wave equation on the static portion (Σ) of the metric, and if γ is the determinant of the spatial part of the metric, then the inner product on the space of square integrable functions, $L^2(\Sigma)$, takes the form

$$(\chi, \zeta) = \int d^3x \, (\gamma)^{\frac{1}{2}} (-\gamma_{tt})^{-\frac{1}{2}} \chi(x^a) \zeta(x^b)$$

Procedure

Rewrite K–G radial equation in
“1D Schrodinger form” $H\Psi(x) = E\Psi(x)$
where the operator
$H = -d^2/dx^2 + V(x)$ and E is a constant
and
any singularity is taken to be at $x = 0$.
Then determine whether $V(x)$ is
LP or LC.

Conformally Static Spacetimes

3 Examples:

1. FRW with Cosmic String

2. Roberts Spacetime

3. A Class of Spherically-Symmetric Self-Similar STs

1. FRW with Cosmic String

Model by Davies and Sahni (1988)

$$ds^2 = a^2(t)\,(-dt^2 + dr^2 + \beta^2 r^2\, d\phi^2 + dz^2)$$

where

$\beta = 1 - 4\mu$ and μ is the mass per unit length of the cosmic string. This metric is conformally static (actually conformally flat).

Classical Singularity Structure

$a(t) = 0$: Scalar curvature singularity

$\beta^2 \neq 1$: Quasiregular singularity

The latter is in the related static metric, it is timelike, and we will investigate its quantum singularity structure.

Quantum Singularity Structure

Klein–Gordon Equation with general coupling:

$$|g|^{-1/2} \left(|g|^{1/2} g^{\alpha\beta} \Phi_{,\beta}\right)_{,\alpha} = (M^2 + \xi R)\, \Phi$$

With mode solutions

$$\Phi \approx T(t)\, H(r)\, e^{im\phi}\, e^{ikz}$$

where the T–eqn alone contains M and R,

Schrodinger Form of Radial Equation

The radial equation is

$$H'' + (1/r)\, H' + (-k^2 - q - (m^2/\beta^2 r^2))\, H = 0.$$

Let $r = x$ and $H = x\, u(x)$ to get correct inner product and 1D Schrodinger Equation,

$$u'' + (E - V(x))u = 0$$

Where $E = -k^2 - q$ and $V(x) = (m^2 - \beta^2/4)/\beta^2 x^2$.

Quantum Singularity Structure

Near zero, $V(x)$ is limit point if $m^2/\beta^2 \geq 1$.

The $m = 0$ mode is clearly limit circle so the singularity is

Quantum Mechanically Singular

2. Roberts Solution

The Roberts (1989) metric is

$$ds^2 = e^{2t} (-dt^2 + dr^2 + G^2(r)\, d\Omega^2)$$

where

$$G^2(r) = \tfrac{1}{4}[\, 1 + p - (1 - p)\, e^{-2r}](e^{2r} - 1).$$

It is conformally static, spherically symmetric and self-similar. Classical scalar curvature singularity at $r = 0$ for $0 < p < 1$ that is timelike.

Quantum Singularity Structure

After separating variables in the massless, generally-coupled Klein-Gordon equation, the potential in the 'Schrodinger'-form of the radial portion takes the form:

As $x \to 0$,

$V(x) \sim (-1 + 2\xi)/4x^2$ for $\xi \neq 1/2$, and

$V(x) \sim (l^2 + l + 1)/px$ for $\xi = 1/2$.

Therefore, Roberts spacetime is LC (Quantum mechanically singular) unless $\xi \geq 2$. In particular, it is LC for both minimally and conformally coupled waves.

3. A Class of Self-Similar Spacetimes

Consider next a class of STs given by Brady (1994),

$$ds^2 = e^{2t} [-g_1(u)\, dt^2 + g_2(u)\, du^2 + e^{2u}\, d\Omega^2]$$

where $d\Omega^2 = d\theta^2 + \sin^2(\theta)\, d\phi^2$.

These are conformally static, self-similar spacetimes where $g_1(u)$ and $g_2(u)$ are arbitrary functions. Here, for simplicity, let $g_1(u)=au^m$ and $g_2(u)=bu^n$. These STs have timelike scalar curvature classical singularities (reachable by null geodesics) for $m \leq n+2$ at $u=0$.

Quantum Singularity Structure

After separating variables in the massless, minimally-coupled Klein-Gordon equation, the potential in the 'Schrodinger-form' of the radial portion takes the form:

$$\Psi'' + \big(E - V(x)\big)\Psi = 0$$

where

$$V(x) \sim A\, x^{\frac{2m}{n+2-m}} + B\, x^{\frac{2(m-n-1)}{n+2-m}}$$

as x→ 0.

And, in general, $V(x) < (3/4)\, x^{-2}$ for all classically singular cases.

Therefore, this class of spacetime is LC (Quantum mechanically singular) in all classically singular cases.

Future Research

We have found that the use of quantum waves is successful in resolving classical singularities in some classes of spacetimes, but not others….

WHY????

A study of more general classes of space-times is underway.

This is still a work in progress. One goal for the future is to understand more comprehensively and more deeply which singularities can be resolved using quantum waves, and which cannot, and why.