The Einstein Flow on closed surfaces

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Consider the following problem:

What is the global geometry of the maximal globally hyperbolic development of a given initial data set (M, g, k) under the Einstein flow, where M is closed without boundary?

Open questions

- How many incomplete/complete time directions ? (recollapse vs. expansion vs. completeness)
- Is there a relation between the spatial topology and the global geometry ? (*Closed Universe Recollapse*)
- Does incompleteness imply curvature blow-up ? (*Strong Cosmic Censorship*)
- Stability/Instability of homogeneous models ?
- How does matter or a cosmological constant affect these questions ?

Results (*symmetry* (effectively 1+1-dimensional) and/or *small data*)

Choquet-Bruhat—Moncrief, Andersson—Moncrief, Anderson, Friedrich, Ringström, Lin—Wald, Rendall, Rendall—Dafermos, Rendall—Burnett, Henkel, Andréasson, Rein, LeFloch-Smulevici, Smulevici, Rodnianski-Speck, … (not exhaustive!)

The corresponding problem in 2+1-dimensions:

What is the global geometry of the MGHD of (M, g, k) under the Einstein flow, where M is a closed, orientable **surface** without boundary?

Motivation:

- A toy-model to eventually address the case of general data without symmetry assumptions.
- Curvature simpler (yet non-trivial) in 2+1-dimensions.
- Trichotomy for the topology: Sphere — Torus — Hyperbolic surfaces.
- Conformal metric determined by evolution in Teichmüller space.
- The *Moncrief U(1)-symmetry reduction:* 2+1 gravity + wave map = U(1)-symmetric 3+1 vacuum gravity.

What is known for the vacuum case ?

- Non-existence of spacetimes with spherical spacelike topology.
- For toroidal spatial topology, the conformal geometry degenerates asymptotically (in explicit models).
- Global existence for hyperbolic surfaces (Andersson/Moncrief/Tromba) (no curvature blow-up).
- Barrow-Burd-Lancaster (`86), Barrow-Shaw-Tsagas (`06)

The non-vacuum Einstein flow on surfaces

Matter resolves the rigidity in 2+1 dimensions and allows for cosmological solutions of **all topologies** and **all types of asymptotic behavior**.

Einstein-Vlasov flow on surfaces:

$$\overline{M} = I \times M,\ M \in \{\mathbb{S}^2, \mathbb{T}^2, \mathbb{H}_k\}$$

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = T_{\mu\nu}(f)$$

$$Xf = 0$$

$$\overline{g} = -N^2 dt^2 + g_{ab} dx^a dx^b,\ g = e^{2\lambda}\sigma,\ R(\sigma) \in \{-1, 0, 1\}$$

$$D_b h^b_a = -e^{2\lambda} j_a$$

$$2\Delta_\sigma \lambda = e^{2\lambda}/2\tau^2 + R(\sigma) - 2e^{2\lambda}\rho - e^{-2\lambda}|h|^2_\sigma$$

$$\Delta_\sigma N = N e^{2\lambda}\left(e^{-4\lambda}|h|^2_\sigma + \tau^2/2 + \eta\right) - \partial_t \tau e^2$$

$$\partial_t h_{ab} = \dots$$

- Only self-gravitating particles, no additional effects.
- Regular on regular backgrounds: Only „meaningful“ singularities.
- Cases of **massive** and **massless particles** can be considered.

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- Cases of **massive** and **massless particles** can be considered.
- The rescaled **energy density decreases** the conformal **scalar curvature**.

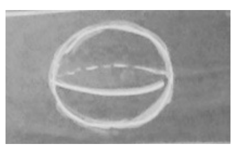
Results

I. MASSIVE PARTICLES

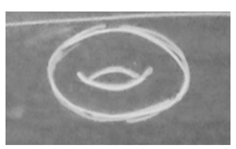
The massive EV flow with sufficiently large total mass yields future complete and past incomplete spacetimes for **each spatial topology**.

$$\mathbf{m}_\infty = \int_M \rho_\infty \mu_g, \quad \rho_\infty = \int f \sqrt{g} dp < \rho$$

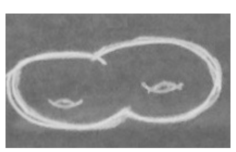
total mass, mass density



$$\tau^2 g_{(+)} \xrightarrow{\tau \uparrow 0} \frac{\mathbf{m}_\infty(\mathbb{S}^2) - 2\pi\chi(\mathbb{S}^2)}{\pi\chi(\mathbb{S}^2)} \cdot \sigma_{\mathbb{S}^2}$$



$$\tau^2 g_{(0)} \xrightarrow{\tau \uparrow 0} \frac{4\mathbf{m}_\infty(\mathbb{T}^2)}{\mathrm{vol}_\sigma(\mathbb{T}^2)} \cdot \sigma_{\mathbb{T}^2}$$



$$\tau^2 g_{(-)} \xrightarrow{\tau \uparrow 0} \frac{\mathbf{m}_\infty(\mathbb{H}_k) - 2\pi\chi(\mathbb{H}_k)}{\pi|\chi(\mathbb{H}_k)|} \cdot \sigma_{\mathbb{H}_k}$$

future asymptotic behavior, homogeneous models

$$\lim_{\tau \to -\infty} R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} = \infty$$

curvature blow-up, inextendibility for homogeneous models

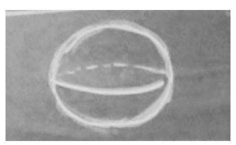
- THEOREM (F. '15) All spacetimes of the above type are **nonlinearly future stable**.
- Spatial topology irrelevant for the long-time behavior of the flow (close to homogeneous models).
- Recollapse conjecture does not apply to massive EV in 2+1-dimensions.

2. MASSLESS PARTICLES

The massless EV flow is **sensitive** to the **spatial topology** —spheres recollapse, tori expand slowly, hyperbolic surfaces retain vacuum behavior.

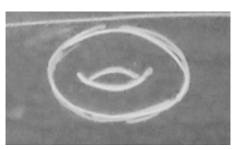
$$\mathbf{C}_{\mathsf{msl}} = e^{3\lambda}\rho, \quad \rho = \int f|p|_g\sqrt{g}dp = \eta$$

rescaled density conserved



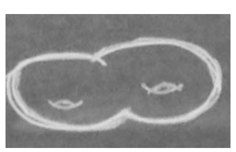
$$(\mathbf{C}_{\mathsf{msl}})^{2/3}\cdot\sigma_{\mathbb{S}^2} \leq |\tau|^{4/3}g_{(+)} \leq (2\mathbf{C}_{\mathsf{msl}})^{2/3}\cdot\sigma_{\mathbb{S}^2}, \quad \tau\in(-\infty,+\infty)\setminus(-1,1)$$
$$\mathrm{vol}_{g_{(+)}(\tau=0)}(\mathbb{S}^2) = \mathrm{vol}_{\sigma_{\mathbb{S}^2}}(\mathbb{S}^2)(2\mathbf{C}_{\mathsf{msl}})^2$$

recollapse



$$|\tau|^{4/3}g_{(0)} = (4\mathbf{C}_{\mathsf{msl}})^{2/3}\cdot\sigma_{\mathbb{T}^2}, \quad \tau\in(-\infty,0)$$

future completeness



$$2\cdot\sigma_{\mathbb{H}_k} < \tau^2 g_{(-)} < (2+\mathsf{const})\cdot\sigma_{\mathbb{H}_k}, \quad \tau\in(-\infty,0)$$

$$\lim_{\tau\to\pm\infty} R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} = \infty$$

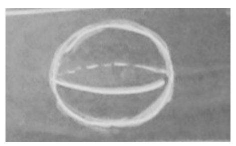
curvature blow-up at all singularities

- Recollapsing behavior for spheres is **stable** by Cauchy stability and singularity theorems (analog to Ringström — The Cauchy problem in General Relativity, '09).
- Models valid for general massless matter. In the homogeneous case massless matter just appears as a constant in the rescaled system.
- A possible **Recollapse conjecture** in 2+1-D: The massless EV-flow yields recollapsing spacetimes.

3. NON-VANISHING COSMOLOGICAL CONSTANT

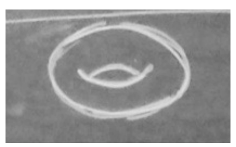
For positive Λ the EV flow is **sensitive to the spatial topology** while for negative Λ **all models recollapse**.

$\Lambda > 0$

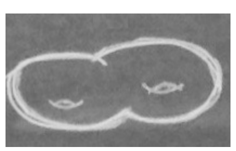


m=0: $54\mathbf{C}^2_{\mathsf{msl}} \cdot \Lambda \leq 1$ recollapse, otherwise past incomplete + future complete

m=1: $2\mathbf{m}_\infty \geq 1$ past incomplete + future complete, otherwise depends on velocity distribution (if small then recollapse)



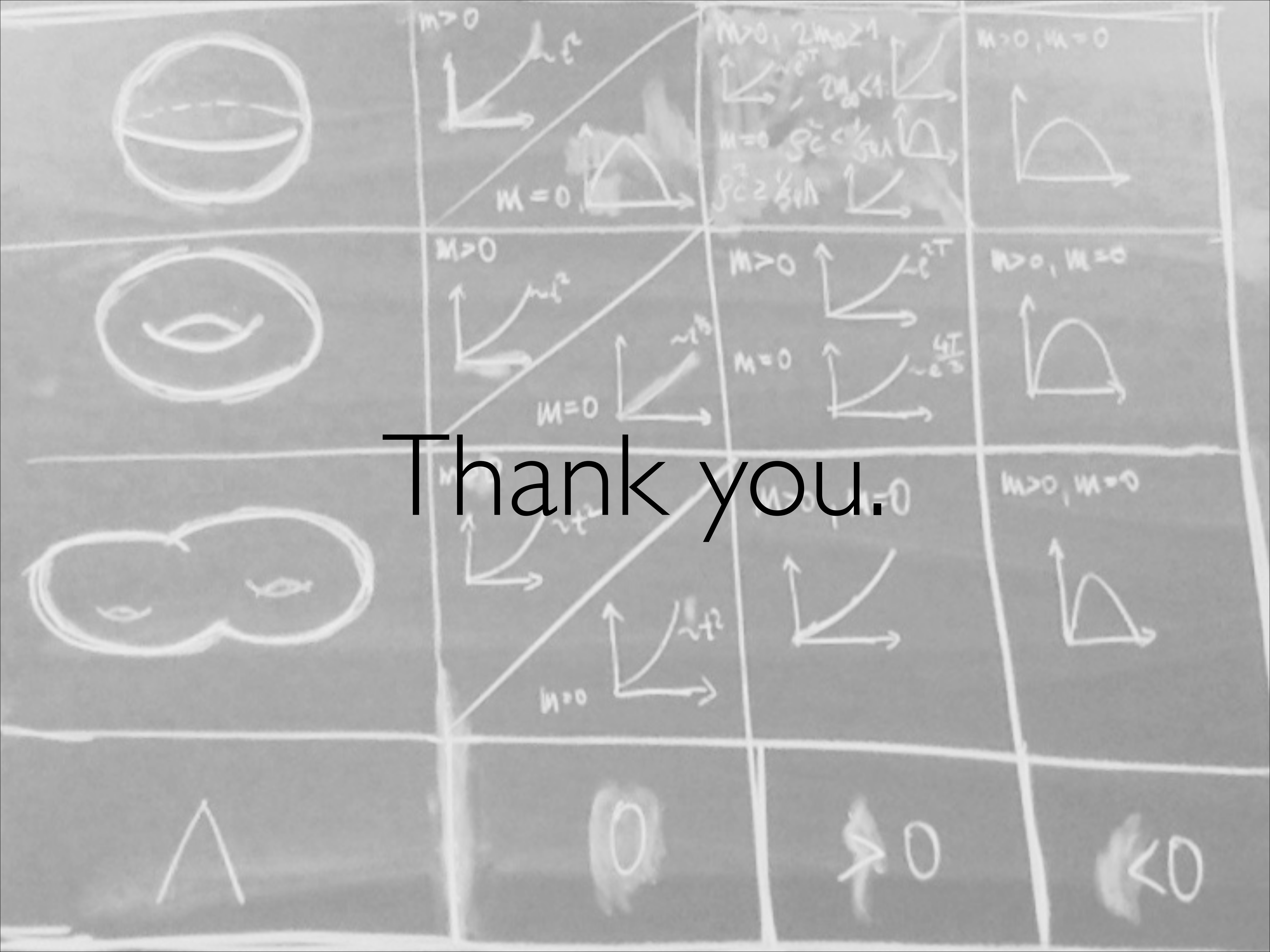
m=0,1 past incomplete + future complete



m=0,1 past incomplete + future complete

$\Lambda < 0$ all models recollapse

- Vacuum solutions with negative Λ and spherical spacelike topology do not exist.
- Certain solutions do not admit global CMC foliations.



Thank you.