Testing gravitomagnetism with clocks

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Outline

Clocks

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The Kerr space-time

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Clock effects in Kerr space-time

- Why also gravitomagnetic clock effect?
- The gravitomagnetic clock effect
- Gravitomagnetic clock effect on Earth

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Further issue

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Summary and outlook

Testing gravitomagnetism with clocks

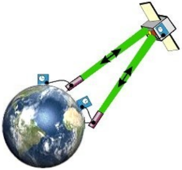
The gravitomagnetic field of the Earth which within General Relativity is related to the rotation of the Earth has been observed by satellite orbits (LAGEOS, LARES) as well as by the precession of a spinning top (GP-B). Being encoced in the space-time metric its gravitomagnetic components should also have an effect on clocks. This has first been observed by Cohen and Mashoon for a pair of satellites counter-orbiting the Earth. We report on a generalization of this effect for arbitrary pairs of satellites [Phys. Rev. D 90, 044059 (2014)]. This may also be used for testing gravitomagnetism using pulsar timing. In addition to that, we will show that clocks on Earth are also sensitive to the gravitomagnetic field. Clock networks which are in the state of being established around the world will be crucial for that. We also can use a comparison of clocks in space with clocks on Earth in order to measure the gravitomagnetic field. For each case we calculate the effect and discuss the measurability.

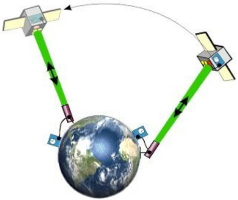
Outline

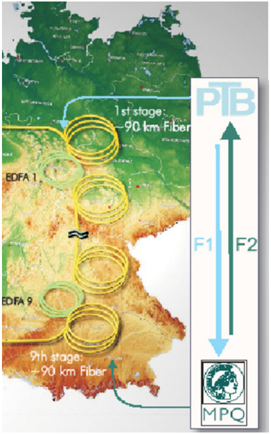
Performance of clocks

Optical clocks and fiber networks achieved the accuracy level to provide differences of the gravity potential corresponding to 1 cm in height

- Optical clocks
  - accuracy $5 \cdot 10^{-18}$ (Bloom et al, Nature 2014)
  - stability $1.6 \cdot 10^{-18}$ (Hinkley et al, Science 2013)
- Clock comparison
  - Optical fibers on Earth: stability $\sim 10^{-19}$ (Droste et al, PRL 2013)
  - Optical links to satellites: stability $\sim 10^{-14} \ldots 10^{-15}$







Relativistic clock comparison effects effects

effect	term	on Earth	for satellites
longitudinal Doppler	v/c	negligible	$\leq 10^{-2}$ per day
transversal Doppler	v^2/c^2	Earth rotation	$\leq 10^{-5}$ s per day
Sagnac effect	$\omega\Omega\Sigma/c^2$	up to 10^{-13}	$\sim 10^{-7}$ s per orbit
1^{st} order grav. redshift	$\Delta U/c^2$	up to 10^{-14}	$\sim 10^{-7}$ s per day
2^{nd} order grav. redshift	$(\Delta U/c^2)^2$	negligible	$\sim 10^{-14}$ s per day
gravitational time delay	$\sim \frac{GM}{c^2} \ln \frac{r_1 r_2}{b^2}$	negligible	$\sim 10^{-11}$ s
gravitomagn. clock effect	J/Mc^2	measurable(?)	$\sim 10^{-7}$ s per orbit

relevant effects have to be included in TAI and in Galileo ($\sim$ 10 km/day)
in general, most of these effects cannot be strictly separated – this is possible only for weak fields

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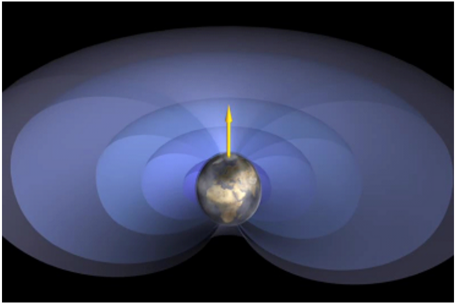
The gravitomagnetic field



leads to

- precession of orbital plane: Lense-Thirring effect (LAGEOS, LARES – Ciufolini)
- precession of classical spin: Schiff effect (GP-B – Everitt et al PRL 2012)
- could be measured by atom interferometry (HYPER)
- precession of quantum spin (Hehl)
- gravitomagnetic clock effect (Cohen & Mashhoon, PLA 1993, Hackmann & C.L., PRD 2014)

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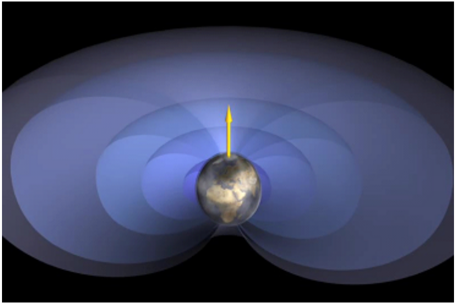

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Outline

The Kerr solution

Einstein equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

solution of Einstein field equation describing a rotating black hole: Kerr metric

$$ds^2 = \frac{\Delta}{\rho^2} \left(dt - a \sin^2\theta d\varphi \right)^2 - \frac{\rho^2}{\Delta} dr^2 - \frac{\sin^2\theta}{\rho^2} \left(a dt - (r^2 + a^2) d\varphi \right)^2 - \rho^2 d\theta^2$$

where

$$\begin{aligned} \rho^2 &= r^2 + a^2 \cos^2\theta \\ \Delta &= r^2 + a^2 - 2Mr \end{aligned}$$

M mass of the black hole, a specific angular momentum

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Geodesics and conservations laws

geodesic equation

$$0 = \frac{d^2 x^\mu}{ds^2} + \left\{ {}^{\;\mu}_{\rho\sigma} \right\} \frac{dx^\rho}{ds} \frac{dx^\sigma}{ds}$$

existence of four constants of motion

- **energy** per unit mass

$$E := g_{tt}\dot{t} + g_{t\varphi}\dot{\varphi}$$

- **angular momentum** per unit mass in z direction

$$L_z := -g_{\varphi\varphi}\dot{\varphi} - g_{t\varphi}\dot{t}$$

  where the dot denotes a derivative with respect to the proper time s

- **normalization** $\epsilon = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$ with $\epsilon = 1$ for timelike and $\epsilon = 0$ for null geodesics
- **Carter constant** $\mathcal{C}$ and K with $\mathcal{C} = K - (aE - L_z)^2$ such that $\mathcal{C} = 0$ or $K = (aE - L_z)^2$ corresponds to motion confined to equatorial plane

Solution of the geodesic equation

geodesic equation equivalent to Hamilton–Jacobi equation

$$2\frac{\partial S}{\partial \tau} = g^{ij}\frac{\partial S}{\partial x^i}\frac{\partial S}{\partial x^j}$$

separation ansatz

$$S = \frac{1}{2}\epsilon s - Et + L_z\varphi + S_r(r) + S_\theta(\theta)$$

- insertion into Hamilton–Jacobi
- separation of r and ϑ equations
- separation constant $= K =$ Carter constant (Carter, PR 1968)
- introduction of Mino time τ through $d\tau = \rho^2 ds$ (Mino, PRD 2003)
- renormalization: all quantities in units of r_S

Solution of the geodesic equation

separation of geodesic equation

$$\left(\frac{dr}{d\tau}\right)^2 = \mathcal{R}^2 - \Delta(\epsilon r^2 + K) =: R(r)$$
$$\left(\frac{d\theta}{d\tau}\right)^2 = K - \epsilon a^2 \cos^2\theta - \frac{\mathcal{T}^2}{\sin^2\theta} =: \Theta(\theta)$$
$$\frac{d\varphi}{d\tau} = \frac{a\mathcal{R}}{\Delta} - \frac{\mathcal{T}}{\sin^2\theta} =: \Phi_r(r) + \Phi_\theta(\theta)$$
$$\frac{dt}{d\tau} = \frac{(r^2+a^2)\mathcal{R}}{\Delta} - a\mathcal{T} =: T_r(r) + T_\theta(\theta)$$

with

$$\mathcal{R} = (r^2+a^2)E - aL_z\,, \qquad \mathcal{T} = aE\sin^2\theta - L_z$$

- equations can be solved completely and explicitly in a closed analytic form by Weierstraß elliptic functions (Hackmann, PhD thesis 2009)
- can be extended to the whole class of Plebański-Demiański black hole space-times (Hackmann, Kagramanova, Kunz, C.L., EJPL 2009) using hyperelliptic functions, and to further space-times

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Why clock effect?

- clock effects are a particular feature of relativity
- LT effect can be mimicked in Newtonian gravity with J_2 term – need clocks in order to distinguish Newton from GR, same holds for Perihelion shift
- in generalized theories of gravity orbital effects may differ from time effects (could also be true for SR: Michelson-Morley vs. Ives-Stilwell)

What is a clock?

A clock is a **standard clock** as defined by V. Perlick (V. Perlick, GRG 1987)

- a standard clock is a particular parametrization of a worldline so that the **measured** relative acceleration of passing freely falling particles does not depend on the **measured** relative velocity of these particles (is just UFF)
- the time of a standard clock is given by $s = \int \sqrt{g_{\mu\nu} dx^\mu dx^\nu}$

in not too strong gravitational fields atomic clocks are an extremely good realization of standard clocks

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Outline

Observables: for bound orbits

For bound orbits

- two oscillatory coordinates: r and ϑ (generalized Lissajous figures)
- two (secularly) increasing coordinates: t and φ

Periods

- radial period

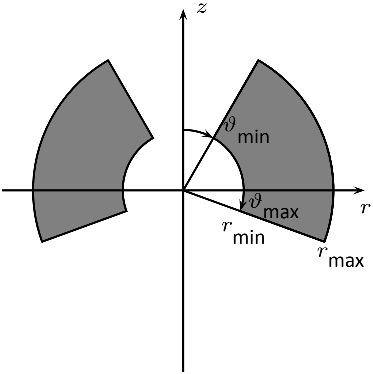
$$\omega_r = 2 \int_{r_{\text{min}}}^{r_{\text{max}}} \frac{dr}{\sqrt{R}}$$

is time needed to go from r_{min} to r_{min}

- polar angle period

$$\omega_\vartheta = 2 \int_{\vartheta_{\text{min}}}^{\vartheta_{\text{max}}} \frac{dr}{\sqrt{\Theta}}$$

is time needed to go from ϑ_{min} to ϑ_{max}



Observables: for bound orbits

Secular increases

- secular time increase

$$Y_t = \left\langle \frac{dt}{d\tau} \right\rangle = \frac{2}{\omega_r} \int_{r_{\mathrm{min}}}^{r_{\mathrm{max}}} h(r) \frac{dr}{\sqrt{R}} + \frac{2}{\omega_\vartheta} \int_{\vartheta_{\mathrm{min}}}^{\vartheta_{\mathrm{max}}} j(\vartheta) \frac{dr}{\sqrt{\Theta}}$$

- secular azimuthal increase

$$Y_\varphi = \left\langle \frac{d\varphi}{d\tau} \right\rangle = \frac{2}{\omega_r} \int_{r_{\mathrm{min}}}^{r_{\mathrm{max}}} f(r) \frac{dr}{\sqrt{R}} + \frac{2}{\omega_\vartheta} \int_{\vartheta_{\mathrm{min}}}^{\vartheta_{\mathrm{max}}} g(\vartheta) \frac{dr}{\sqrt{\Theta}}$$

orbital frequencies (Drasco & Hughes, PRD 2004; Schmidt, CQG 2004)

$$\Omega_r = \frac{2\pi}{Y_t \omega_r}, \qquad \Omega_\vartheta = \frac{2\pi}{Y_t \omega_\vartheta}, \qquad \Omega_\varphi = \frac{Y_\varphi}{Y_t}$$

- angular velocity of r–oscillations
- angular velocity of ϑ–oscillations
- secular azimuthal angular velocity

Observables: for bound orbits

observables: self referential comparison, invariant

The observables

- periastron shift

$$\Delta_{\text{periastron}} := \Omega_\varphi - \Omega_r = \left(Y_\varphi - \frac{2\pi}{\omega_r} \right) \frac{1}{Y_t}$$

- Lense–Thirring effect

$$\Delta_{\text{Lense-Thirring}} := \Omega_\varphi - \Omega_\vartheta = \left(Y_\varphi - \frac{2\pi}{\omega_\vartheta} \right) \frac{1}{Y_t}$$

explicit evaluation by complete hyperelliptic integrals in all Plebański-Demiański space-times (Hackmann & C.L. PRD 2012)

Fundamental frequencies

from averaged quantities: coordinate time for a revolution of 2π

$$t(2\pi) = 2\pi \frac{Y_t}{Y_\varphi}$$

- Newtonian limit: Kepler law $t(2\pi) = 2\pi\sqrt{\frac{a^3}{Gm}}$
- with $J = 0$ the Schwarzschild expression is recovered given in Hackmann & C.L., PRD 2012

new definition of a **fundamental frequency for the proper time**

$$Y_s := \frac{2}{\omega_r}\int_{r_p}^{r_a} \frac{r^2 dr}{\sqrt{R(r)}} + \frac{2}{\omega_\theta}\int_{\theta_{\text{min}}}^{\theta_{\text{max}}} \frac{J^2 \cos^2\theta d\theta}{\sqrt{\Theta(\theta)}}$$

gives proper time for a revolution of 2π

$$s(2\pi) = 2\pi \frac{Y_s}{Y_\varphi}$$

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Special gravitomagnetic clock effect

- clocks in counterpropagating circular orbits (Cohen and Mashhoon 1993)

$$s_+ - s_- \approx 4\pi \frac{J}{Mc^2}$$

  where s_+ is the proper time of the corotating and s_- of the counterrotating clock
- for satellites orbiting the Earth $\Delta s \approx 10^{-7}\, s$ per revolution
- generalization to inclined eccentric orbits (Mashhoon et al 2001)

$$s_+ - s_- \approx 4\pi \frac{J \cos i}{Mc^2} \left[\frac{-3}{\sqrt{1-e^2}} + \frac{4 - 2\cos^2\varphi_0 \tan^2 i}{(1 + e\cos(\varphi_0 - g))^2} \right]$$

  i inclination, e eccentricity, g argument of the pericenter, $v_0 = \varphi_0 - g_0$ is the true anomaly at $t = 0$

we want to compare the proper times of arbitrary orbits ... $\rightarrow$

General gravitomagnetic clock effect

observable for the gravitomagnetic clock effect for arbitrary orbits

New observable (Hackmann & CL, PRD 2014)

for two geodesics in Kerr space-time: define a new observable

$$\Delta s_{\text{gm}}(J) := s_1(\pm 2\pi; J) + \alpha\, s_2(\pm 2\pi; J)$$

α is given such that the usual gravitoelectric effects cancel

- α can be calculated from orbital data: compute energies E_n, angular momenta $L_{z,n}$, Carter constants K_n (all depend on angular momentum J)
- determine $E_n(0)$, $L_{z,n}(0)$, and $K_n(0)$ by setting $J = 0$ and $s_n(2\pi; 0)$. α can be determined through $\Delta s_{\text{gm}} = 0$ for $J = 0$

$$0 = \Delta s_{\text{gm}}(J = 0) = s_1(\pm 2\pi; 0) + \alpha s_2(\pm 2\pi; 0)$$

and, therefore,

$$\alpha = -\frac{s_1(\pm 2\pi; 0)}{s_2(\pm 2\pi; 0)}$$

The gravitomagnetic clock effect on satellites

special case: for counterpropagating circular motion

- $\alpha = -1$
- effect to first order in J

$$\begin{aligned}\Delta s_{\text{gm}} &= s_+(2\pi) - s_-(-2\pi) \\ &= 4\pi J\sqrt{\frac{r}{r-3}} + \mathcal{O}(J^2) \\ &\approx 4\pi J\end{aligned}$$

  for large r
- gives the old result in dimensionless quantities

Application to GNSS satellites

it is possible to measure the gravitomagnetic clock effect using two satellites in arbitrary geodesic orbits with clocks with stabilities of about 10^{-14} over 10^4 s

- Galileo: circular orbit with radius $r_G = 29593$ km, inclination 56° from equatorial plane
- COMPASS: circular orbit with radius of $r_C = 42157$ km, inclination $\sim 0^\circ$ from equatorial plane

then

$$\alpha \approx -0.588$$

gravitomagnetic clock effect

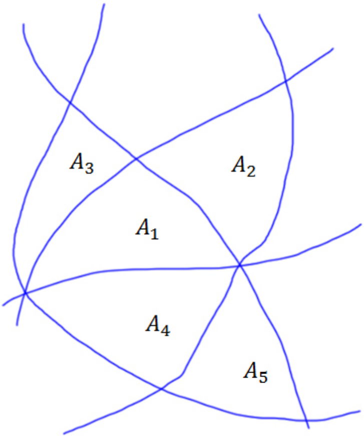
$$\Delta s_{\text{gm}} = s_G(2\pi; J_{\text{Earth}}) + \alpha s_C(2\pi; J_{\text{Earth}}) \approx -7.46 \times 10^{-8}\, s$$

- analysis should be generalized to real satellite orbits: complex gravitational field of Earth should be taken into account
- discussion of requirements on tracking of satellites ($\sim$ 3 cm) and the deviation from ideal orbit (Mashhoon et al 1999)
- signals between satellites
- can also be applied to pulsar timing, with two pulsars orbiting a black hole

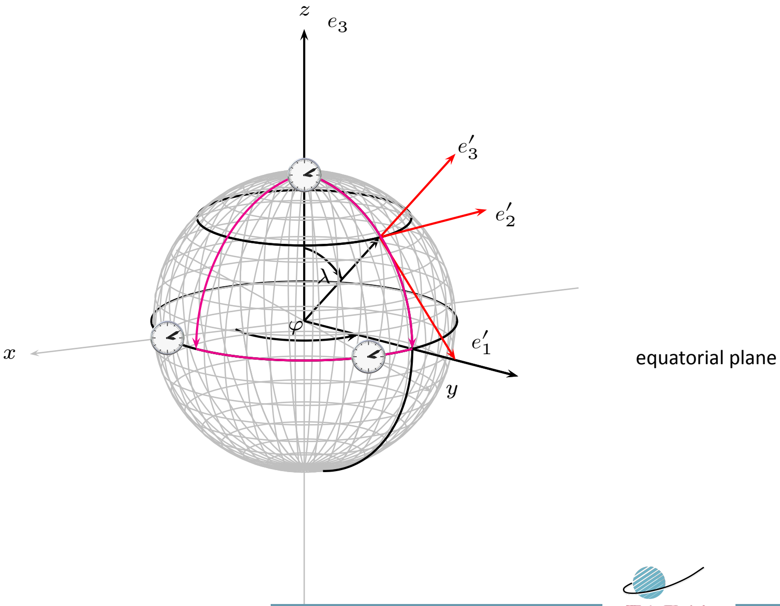
Outline

Clock networks

- fully relativistic description of clocks taking into account all relativistic effects as far as necessary
- description of clock networks
- analysis of stability resp. error sources (e.g. geometry) (see, e.g., Schiller, PRA 2013)
- analyzing advantages/ disadvantages of different geometries for various purposes
- geodetic modeling of time-dependent variations at clock sites
- may be extended to quantum networks (e.g. Komar et al, Nature 2014)



A particular setup



Clocks on Earth

metric of the rotating Earth modelled through Kerr metric
metric on rotating Earth: $r = R$ and $\varphi \to \varphi - \Omega t$

$$\begin{aligned} ds^2 &= \frac{\Delta}{\rho^2}\left(dt - a\sin^2\vartheta(d\varphi - \Omega dt)\right)^2 - \rho^2 d\vartheta^2 \\ &\quad - \frac{\sin^2\vartheta}{\rho^2}\left(adt - \left(r^2 + a^2\right)(d\varphi - \Omega dt)\right)^2 \end{aligned}$$

- clock on North pole ($\vartheta = 0$)

$$ds^2_{\text{pole}} = \frac{\Delta}{\rho^2} dt^2 = \frac{R^2 + a^2 - 2MR}{R^2 + a^2} dt^2$$

- clock on Equator ($\vartheta = \frac{\pi}{2}$)

$$ds^2_{\text{equator}} = \left(1 - \frac{2M}{R}\right) dt^2 + \frac{4aM}{R}\Omega dt^2 - R^2\Omega^2 dt^2 - a^2\left(1 + \frac{2M}{R}\right)\Omega^2 dt^2 \,.$$

Clocks on Earth

clock comparison

$$
\begin{aligned}
ds_{\text{pole}} - ds_{\text{equator}} &= \left(\sqrt{1 - \frac{2MR}{R^2 + a^2}} \right. \\
&\quad \left. - \sqrt{1 - \frac{2M}{R} - \left(R^2 + a^2 + \frac{2Ma^2}{R} \right) \Omega^2 + 4 \frac{Ma}{R} \Omega} \right) dt \\
&= \left(\sqrt{1 - \frac{2M}{R}} - \sqrt{1 - \frac{2M}{R} - \Omega^2 R^2} \right. \\
&\quad \left. - 2 \frac{Ma\Omega}{R\sqrt{1 - \frac{2M}{R} - \Omega^2 R^2}} + O(a^2) \right) dt
\end{aligned}
$$

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Clocks on Earth

essential part is

$$ds_a = \frac{2GM}{Rc^2} a\Omega dt$$

with $a \sim J/M$ where J is the angular momentum of the Earth $= \frac{2}{5} M R^2 \Omega$ ($a_{\text{Earth}} = 3.9$ m $> M_{\text{Earth}} = 4.5$ mm $= r_{S,\text{Earth}}/2$) we obtain

$$ds_a = \frac{2GJ}{Rc^2} \Omega dt$$

and

$$ds_a = \frac{r_S}{R} \frac{2}{5} \left(\frac{R\Omega}{c} \right)^2 dt \approx 5 \cdot 10^{-21} dt$$

for one day $\quad \Delta s \approx 4 \cdot 10^{-16}$ s

redshift $\quad ds = \frac{r_S}{R} \frac{\delta R}{R} dt \sim 2 \cdot 10^{-18}\, dt \quad$ for $\delta R = 1$ cm

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Outline

Relativistic geodesy

general issue: clock effects in the field of the rotating Earth for geodesy

first assumption: stationary situation (but: geodesy looks for mass transport (hydrology, ice melting, sea level))

- exists timelike Killing vector ξ (includes rotation)
- this defines a gravitoinertial potential $\phi = \frac{1}{2} \ln g(\xi, \xi)$
- redshift of clocks given by ϕ, falling bodies or the weight of bodies give the acceleration $-\partial_\mu \phi$

measurements: three types

- chronometry: potential $\phi - \phi_0$ can be probed by clocks
- gravimetry: measuring $-m\partial_\mu \phi$ through falling bodies
- gradiometry: with satellites it is possible to measure the curvature $R^\mu{}_{\nu\rho\sigma}$ (Puetzfeld, Obukhov 2015, Ciufolini, Demianski 1987, Audretsch, C.L. 1982) curvature in our case $\sim \partial_\mu \partial_\nu \phi + \ldots$ (gravity gradient)

Relativistic geodesy

aims: from that one has to derive

- level system (height system) from ϕ
- reference system
- in particular: definition of the reference ellipsoid
- the space-time metric
- the mass distribution of the Earth: requires additional information and modeling, additional challenges in relativistic framework

Outline

Summary and outlook

Summary

- increased stability of clocks requires to discuss all clock effects
- may have impact on clock geodesy, positioning, and navigation
- can be used for improved tests of GR (see analysis of Galileo 5 + 6 $\longrightarrow$ Rievers' talk)

Open issues

- analysis of more general clock network geometries
- analytic clock comparison between clocks on Earth and clocks on satellites: timing
- clock effects in the realistic field of the Earth

Thank you for your attention

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