Mode stability on the real axis in Kerr

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Outline

1. Background
   - Teukolsky Equation
2. Proof
   - Theorem
   - General Strategy
   - Technical Details

Context

- The full nonlinear stability of black holes is one of the big open problems in general relativity.
- Important steps on the way to solving black hole stability are, to obtain dispersive estimates for various test fields on a fixed background and eventually for the linearized Einstein equations.
- Whiting's (1989) proof of mode stability for Kerr was an important step in the question of black hole stability. It states that under the condition of no incoming radiation, no exponentially growing modes can exist.
- We extent this result to the case of real frequencies. This gives us a representation formula for the solution of the inhomogeneous equation which can be used to handle superradiant frequencies in a proof of dispersive estimates.

Teukolsky Master Equation

- The Teukolsky master equation (TME) governs all spin s perturbations on a fix Kerr background geometry.
- For the stability problem it is important to note that the Teukolsky equation does not provide a conserved energy.
- Solutions of the form

$$\Psi_s(r, t, \theta, \phi) = e^{-i\omega t} e^{im\phi}{}_{lm}S_s(\theta)_{lm}R_s(r) \tag{1}$$

  are called mode solutions.
- Mode solutions have infinite energy.

Separated Equations

The radial and angular Teukolsky equation can be written as

$$\mathbf{R}R = 0 \tag{2a}$$

$$\mathbf{S}S = 0 \tag{2b}$$

where the operators are given by

$$\mathbf{R} = \left(\partial_r \Delta \partial_r + \frac{K^2 - 2iK(r-M)s - (r-M)^2 s^2}{\Delta} + 4sir\omega - \Lambda\right) \tag{3}$$

$$\begin{aligned}\mathbf{S} = \frac{1}{\sin\theta}\partial_\theta \sin\theta \partial_\theta - \frac{m^2}{\sin^2\theta} + a^2\cos\theta^2\omega^2 - 2a\omega s\cos\theta \\ - \frac{2ms\cos\theta}{\sin^2\theta} - s^2\cot^2\theta + \Lambda + 2a\omega m - a^2\omega^2,\end{aligned} \tag{4}$$

with $K = (r^2 + a^2)\omega - am$

Superradiance

If we take a scattering ansatz

$$R_s(r) \sim \begin{cases} Y_{\text{hole},s} \Delta^{-s/2} e^{-ik_+ r_*}, & \text{at } r = r_+ \\ Y_{\text{in},s} e^{-i\omega r_*} r^{s-1} + Y_{\text{out},s} e^{i\omega r_*} r^{-s-1}, & \text{at } r = \infty. \end{cases} \tag{5}$$

we can show that the amplitudes have to satisfy the following relation

$$A_s |Y_{\text{hole},s}|^2 = |Y_{\text{in},s}|^2 - B_s |Y_{\text{out},s}|^2 \tag{6}$$

where $B_s > 0$ for all ω and $A_s < 0$ only if $\omega(\omega - am/2Mr_+) < 0$.
Deviding by $|Y_{\text{in},s}|^2$ we get that

$$\mathscr{T} = 1 - \mathscr{R} \tag{7}$$

where $\mathscr{T}$ is the transmission coefficient and $\mathscr{R}$ is the reflection coefficient

Mode Stability on the real axis

Theorem

Let Φ be a separated solution to the TME for the sub-extreme Kerr black hole. Assume that Φ has purely ingoing radiation at the horizon and purely outgoing radiation at infinity. Then $\Phi = 0$.

Hence a solution to the radial Teukolsky equation with asymptotics

$$R \sim \begin{cases} e^{i\omega r} r^{2iM\omega} r^{-s-1} & \text{as } r \to \infty \\ (r - r_+)^{\xi - s/2} & \text{as } r \to r_+ \end{cases} \tag{8}$$

has to be zero.

General Strategy

- Rescaling to canonical form. (Confluent Heun equation)
- Use integral transform to obtain a solution to an equation with different parameters.
- Rescale this new equation to Schrödinger form.
- Show that this Schrödinger equation has a real potential and no superradiance.
- Show that the transformed function is still purely ingoing on the horizon and outgoing at infinity.
- Conclude that the transformed solution is zero and show that this is sufficient to proof that the original solution has to be zero.

Heun equation

The Heun equation for the rescaled radial function $g(r) = (r - r_+)^{-\xi + s/2}(r - r_-)^{-\eta + s/2} e^{i\omega r} R(r)$ is

$$0 = T_r g(r) = (r - r_-)(r - r_+)\frac{d^2 g}{dr^2}$$
$$+ (\gamma(r - r_+) + \delta(r - r_-) + p(r - r_-)(r - r_+))\frac{dg}{dr} + (\alpha p(r - r_-) + \sigma)g \quad (9)$$

with the parameters

$$\gamma = 2\eta + 1 - s$$
$$\delta = 2\xi + 1 - s$$
$$p = -2i\omega$$
$$\alpha = 1 - 2s$$
$$\sigma = -\Lambda - 2i\omega(1 - 2s)r_- - s$$

Kernel for Integral Transform

Let $f(x,r)$ be defined as

$$f(x,r) = e^{-p\frac{(x-r_-)(r-r_-)}{r_+-r_-}}. \tag{10}$$

then requiring

$$(\tilde{T}_x - T_r)f(x,r) = 0. \tag{11}$$

gives us the following relationship between the parameters for $\tilde{T}_x$ and T_r

$$\begin{aligned}
\tilde{\gamma} &:= \alpha \\
\tilde{\delta} &:= \gamma + \delta - \alpha \\
\tilde{p} &:= p \\
\tilde{\alpha} &:= \gamma \\
\tilde{\sigma} &:= \sigma
\end{aligned}$$

Integral Transform

Defining $\tilde{g}(x)$ by

$$\tilde{g}(x) = \int_{\mathscr{C}} f(x,r)(r-r_-)^{\gamma-1}(r-r_+)^{\delta-1} e^{pr} g(r) dr, \tag{13}$$

we have that

$$\begin{aligned}\tilde{T}_x \tilde{g}(x) &= \int_{\mathscr{C}} \tilde{T}_x f(x,r)(r-r_-)^{\gamma-1}(r-r_+)^{\delta-1} e^{pr} g(r) dr \\ &= \int_{\mathscr{C}} T_r f(x,r)(r-r_-)^{\gamma-1}(r-r_+)^{\delta-1} e^{pr} g(r) dr \\ &= \left\{ (r-r_-)^{\gamma}(r-r_+)^{\delta} e^{pr} \left(\frac{df(x,r)}{dr} g(r) - f(x,r)\frac{dg(r)}{dr} \right) \right\}_{\mathscr{C}} \\ &+ \int_{\mathscr{C}} f(x,r)(r-r_-)^{\gamma-1}(r-r_+)^{\delta-1} e^{pr} T_r g(r) dr.\end{aligned} \tag{14}$$

Scaling Heun to Schrödinger

The rescaled function

$$U(x) = (x^2 + a^2)^{\frac{1}{2}}(x - r_-)^{-s}(x - r_+)^{-2iMw} e^{-i\omega x} \tilde{g}(x)$$

then satisfies the Schrödinger equation

$$U'' + V(x)U = 0 \tag{15}$$

where $'$ denotes a derivative with respect to x^* and $\frac{dx^*}{dx} = \frac{x^2+a^2}{\Delta}$.

The important fact is that V is real, hence $(W[U, \bar{U}])' = 0$, and that

$$V\Big|_{x=r_+} = \frac{(r_+ - r_-)^2}{r_+^2}\omega^2, \quad \lim_{x\to\infty} V(x) = \omega^2$$

A detailed analysis of the Integrals gives us the following relations between the limits of the Transformed function and the original function. For the Hankel contour we get

$$\lim_{x\to\infty} |U(x)| = \left(\frac{r_+ - r_-}{2\omega}\right)^{-s+1} \frac{2\pi}{|\Gamma(s-2\xi)|} |r_+ - r_-|^{-s} \lim_{r\to r_+} \left(\Delta^{s/2}|R(r)|\right) \tag{16}$$

For the Euler contour we get

$$\lim_{x\to\infty} |U(x)| = \left(\frac{r_+ - r_-}{2\omega}\right)^{-s+1} |\Gamma(2\xi - s + 1)| |r_+ - r_-|^{-s} \lim_{r\to r_+} \left(\Delta^{s/2}|R(r)|\right) \tag{17}$$