QUASINORMAL MODES OF BLACK HOLES

Claude Warnick

Imperial College, London

21^{st} International Conference on
General Relativity and Gravitation.

New York, 2016

Introduction

- What are quasinormal modes?
- What can they tell us about a spacetime?
- What can one say about specific spacetimes?

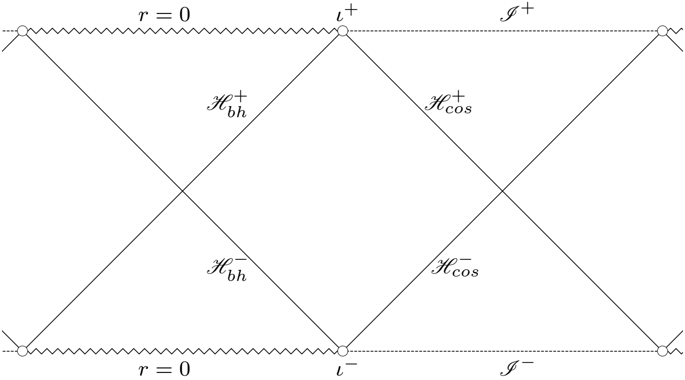
"...we may expect that any intial perturbation will, during its last stages, decay in a manner characteristic of the black hole itself and independent of the cause. In other words, we may expect that during these last stages, the black hole emits gravitational waves with frequencies and rates of damping that are characteristic of the black hole itself, in the manner of a bell sounding its last dying notes."

Chandrasekhar, 1982

Introduction

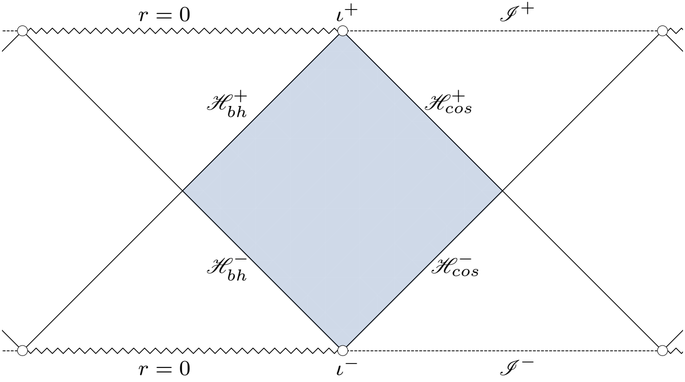
- Black holes settle down to equilibrium by producing radiation at fixed (complex) frequencies: *quasinormal ringdown*
- Frequencies are characteristic of the spacetime, carry geometric information
- Recently there has been a lot of work by mathematicians to understand this phenomenon
- Literature on quasinormal modes is large: I will focus only on a portion
  - See [Kokkotas–Schmidt '99; Konoplya–Zhidenko '11]
- Restrict attention to $\Lambda \neq 0$

Schwarzschild de Sitter



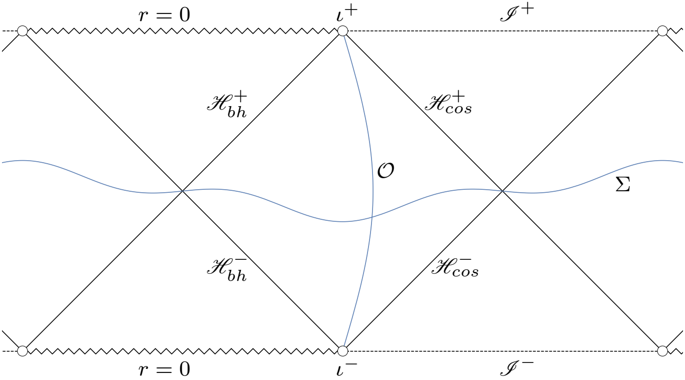
Schwarzschild de Sitter

Schwarzschild de Sitter



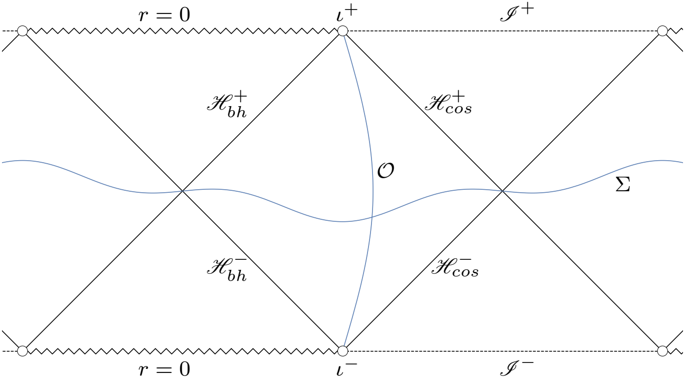
Schwarzschild de Sitter

Schwarzschild de Sitter



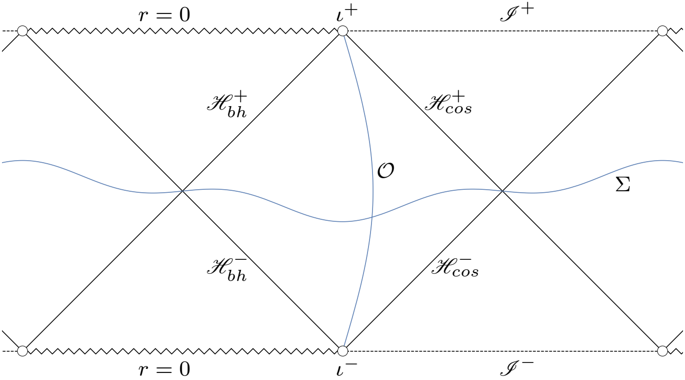
Solve **Einstein's equations** with initial data given on Σ.
What does $\mathcal{O}$ observe at late times?

Schwarzschild de Sitter



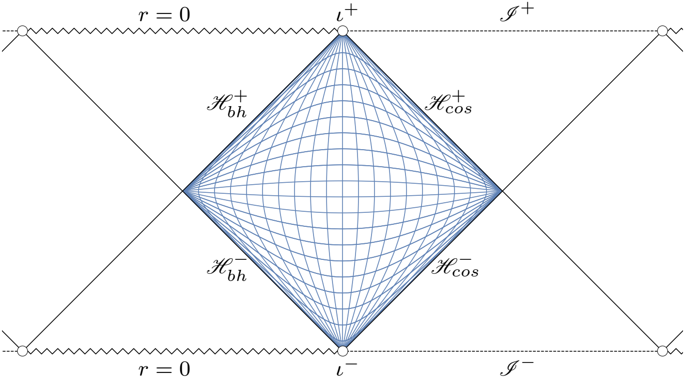
Solve **linearised Einstein's equations** with initial data given on Σ.
What does $\mathcal{O}$ observe at late times?

Schwarzschild de Sitter



Solve $\Box\psi = \mathbf{0}$ with initial data given on Σ.
What does $\mathcal{O}$ observe at late times?

Schwarzschild de Sitter



$$g = -\left(1 - \frac{2m}{r} - \frac{r^2}{l^2}\right) dt^2 + \frac{dr^2}{1 - \frac{2m}{r} - \frac{r^2}{l^2}} + r^2 d\Omega_{S^2}$$

Usual definition

- Separate variables:

$$\psi(t, r, \theta, \varphi) = e^{st} Y_{lm}(\theta, \varphi) R_{sl}(r)$$

satisfies $\Box\psi = 0$ iff:

$$-\frac{d^2}{dr_*^2} R_{sl} + \left[s^2 + V_l\right] R_{sl} = 0, \qquad -\infty < r_* < \infty$$

- The potential V_l decays exponentially as $|r_*| \to \infty$.
- Seek a solution satisfying *outgoing* boundary conditions:

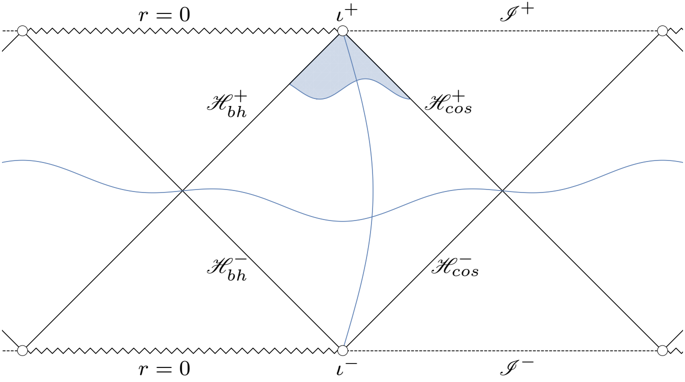
$$R \sim \begin{cases} e^{-sr_*} & r_* \to \infty \\ e^{sr_*} & r_* \to -\infty \end{cases}$$

- Such solutions only occur for a discrete set of $s \in \{\Re z \leq 0\}$, the *quasinormal frequencies*.

Comments

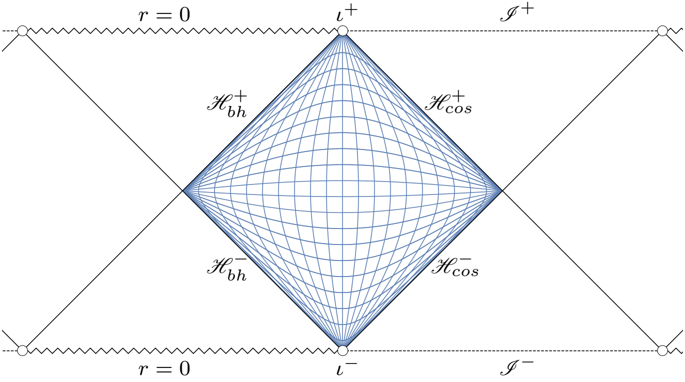
- Expect ψ at late times to decompose as a sum of quasinormal modes
- Difficult in practice to set exponentially decaying branch to zero.
- Looks like an eigenvalue problem, but it isn't!
- Relies on separability of wave equation.
  - Not always possible, particularly for AdS black holes.
- Can make rigorous, but requires an analytic continuation argument: somewhat mysterious!

Schwarzschild de Sitter



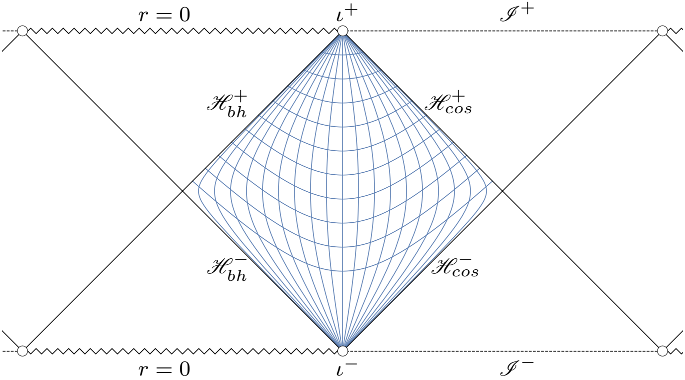
Region of interest

Schwarzschild de Sitter



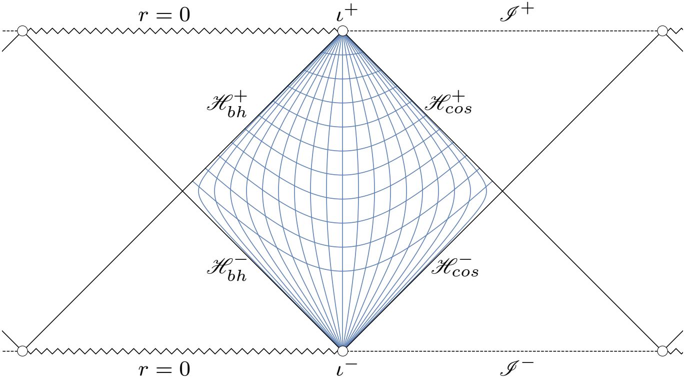
$$g = -\left(1 - \frac{2m}{r} - \frac{\Lambda}{3} r^2\right) dt^2 + \frac{dr^2}{1 - \frac{2m}{r} - \frac{\Lambda}{3} r^2} + r^2 d\Omega_{S^2}$$

Schwarzschild de Sitter



Pick a surface Σ_0 and push forward by the timelike isometry to give a foliation $\{\Sigma_\tau\}_{\tau \geq 0}$

Schwarzschild de Sitter



cf hyperboloidal foliations in AF context

[LeFloch, Friedrich, Donninger, ...]

Semigroups and QNM

- For $k \in \mathbb{N}$, we define a Hilbert space:

$$H^k(\Sigma) = \left\{ (\psi_0, \psi_1) : \Sigma^2 \to \mathbb{C}^2 \;\middle|\; \int_\Sigma \left(\sum_{i \leq k} \left| \partial^i \psi_0 \right|^2 + \sum_{i \leq k-1} \left| \partial^i \psi_1 \right|^2 \right) dS < \infty \right\}$$

Semigroups and QNM

- For $k \in \mathbb{N}$, we define a Hilbert space:

$$H^k(\Sigma) = \left\{ (\psi_0, \psi_1) : \Sigma^2 \to \mathbb{C}^2 \,\middle|\, \int_\Sigma \left(\sum_{i \le k} \left| \partial^i \psi_0 \right|^2 + \sum_{i \le k-1} \left| \partial^i \psi_1 \right|^2 \right) dS < \infty \right\}$$

- Evolution under the wave equation defines a map $\mathcal{S}[\tau] : H^k(\Sigma) \to H^k(\Sigma)$ by:

$$\mathcal{S}[\tau](\psi_0, \psi_1) = (\psi, \partial_\tau \psi)|_{\Sigma_\tau}$$

where ψ is the solution of:

$$\Box \psi = 0, \qquad \psi|_{\Sigma_0} = \psi_0, \quad \partial_\tau \psi|_{\Sigma_0} = \psi_1.$$

Semigroups and QNM

- For $k \in \mathbb{N}$, we define a Hilbert space:

$$H^k(\Sigma) = \left\{ (\psi_0, \psi_1) : \Sigma^2 \to \mathbb{C}^2 \,\middle|\, \int_\Sigma \left(\sum_{i \leq k} \left| \partial^i \psi_0 \right|^2 + \sum_{i \leq k-1} \left| \partial^i \psi_1 \right|^2 \right) dS < \infty \right\}$$

- Evolution under the wave equation defines a map $\mathcal{S}[\tau] : H^k(\Sigma) \to H^k(\Sigma)$ by:

$$\mathcal{S}[\tau](\psi_0, \psi_1) = (\psi, \partial_\tau \psi)|_{\Sigma_\tau}$$

where ψ is the solution of:

$$\Box \psi = 0, \qquad \psi|_{\Sigma_0} = \psi_0, \quad \partial_\tau \psi|_{\Sigma_0} = \psi_1.$$

- $\mathcal{S}[\tau]$ defines a *semigroup* on $H^k(\Sigma)$, which can be written:

$$\mathcal{S}[\tau] = e^{\tau \mathcal{A}}$$

for an unbounded operator $\mathcal{A}$ called the generator.

Semigroups and QNM

Theorem (Vasy '10; Warnick '13; Gannot '16)

For any non-extremal stationary de Sitter or anti de Sitter black hole, $\mathcal{A}$ has a pure point spectrum Λ_k in the half-plane $\Re s > -\varkappa\left(k - \frac{1}{2}\right)$, where $\varkappa$ is the surface gravity. There are countably many eigenvalues, which do not accumulate at any point.

- Get a representation formula for solutions:

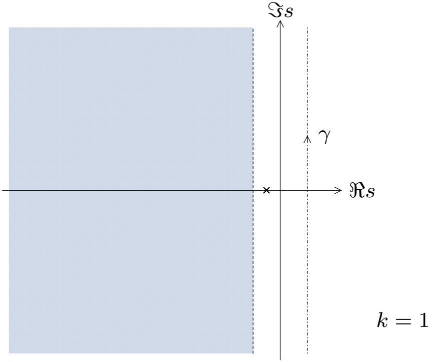
$$\mathbf{\Psi}(t) = \int_\gamma e^{z\tau} \left(\mathcal{A} - z\right)^{-1} \mathbf{\Psi_0} dz$$

- No need to separate variables or perform analytic continuation
- Same methods immediately apply to Dirac, Maxwell, grav. perturbations etc.
- In the case of AdS black holes, boundary conditions are required at $\mathscr{I}$.
- Proof crucially makes use of *redshift effect* [Dafermos–Rodnianski '05]

Semigroups and QNM

Theorem (Vasy '10; Warnick '13; Gannot '16)

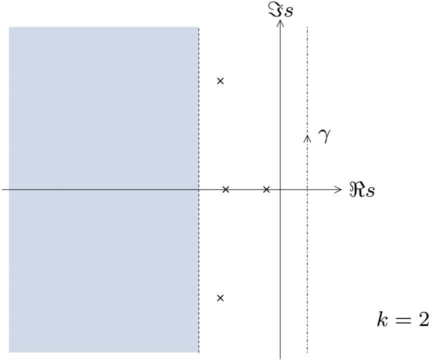
For any non-extremal stationary de Sitter or anti de Sitter black hole, $\mathcal{A}$ has a pure point spectrum Λ_k in the half-plane $\Re s > -\varkappa\left(k - \frac{1}{2}\right)$, where $\varkappa$ is the surface gravity. There are countably many eigenvalues, which do not accumulate at any point.



Semigroups and QNM

Theorem (Vasy '10; Warnick '13; Gannot '16)

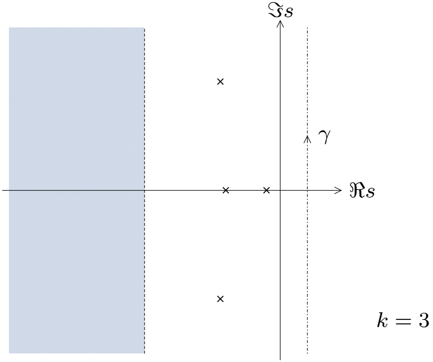
For any non-extremal stationary de Sitter or anti de Sitter black hole, $\mathcal{A}$ has a pure point spectrum Λ_k in the half-plane $\Re s > -\varkappa\left(k - \frac{1}{2}\right)$, where $\varkappa$ is the surface gravity. There are countably many eigenvalues, which do not accumulate at any point.



Semigroups and QNM

Theorem (Vasy '10; Warnick '13; Gannot '16)

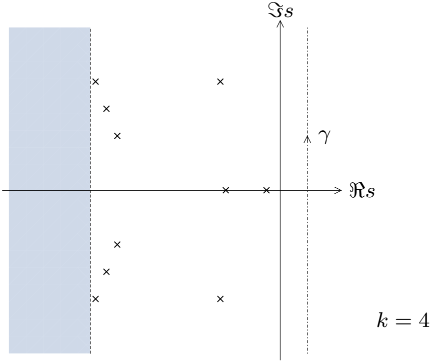
For any non-extremal stationary de Sitter or anti de Sitter black hole, $\mathcal{A}$ has a pure point spectrum Λ_k in the half-plane $\Re s > -\varkappa\left(k - \frac{1}{2}\right)$, where $\varkappa$ is the surface gravity. There are countably many eigenvalues, which do not accumulate at any point.



Semigroups and QNM

Theorem (Vasy '10; Warnick '13; Gannot '16)

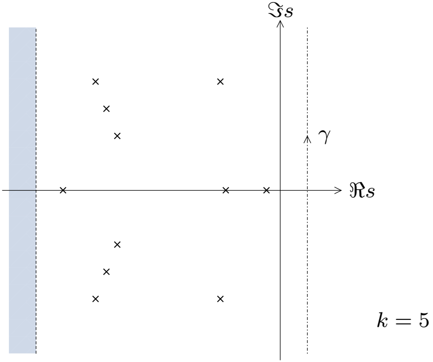
For any non-extremal stationary de Sitter or anti de Sitter black hole, $\mathcal{A}$ has a pure point spectrum Λ_k in the half-plane $\Re s > -\varkappa\left(k - \frac{1}{2}\right)$, where $\varkappa$ is the surface gravity. There are countably many eigenvalues, which do not accumulate at any point.



Semigroups and QNM

Theorem (Vasy '10; Warnick '13; Gannot '16)

For any non-extremal stationary de Sitter or anti de Sitter black hole, $\mathcal{A}$ has a pure point spectrum Λ_k in the half-plane $\Re s > -\varkappa\left(k - \frac{1}{2}\right)$, where $\varkappa$ is the surface gravity. There are countably many eigenvalues, which do not accumulate at any point.



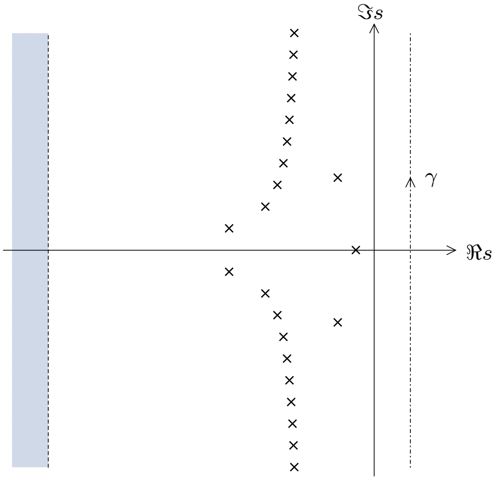
Spectral gaps and expansions

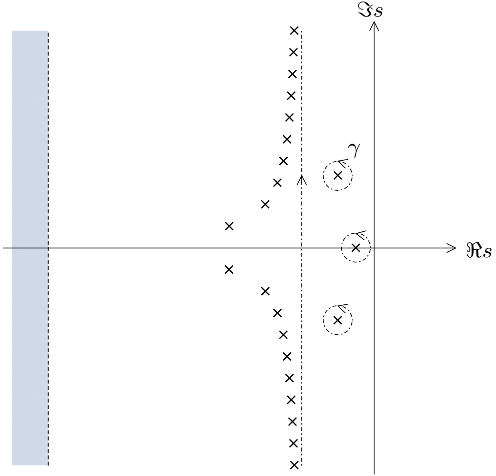
- We can try and shift the contour in the representation formula above.
- If this is possible, get an *asymptotic expansion*: for some $a_i \in \mathbb{C}$

$$\psi(x,\tau) = \sum_{s_i \in \Lambda, \Re s_i > -\nu} a_i u_i(x) e^{s_i \tau} + \mathcal{O}\left(e^{-\nu\tau}\right)$$

- Fundamental obstruction to shifting the contour: *trapped null geodesics*.
- Provided trapping is *normally hyperbolic* (e.g. photon sphere), can shift the contour. [Vasy '10; Dyatlov '14]
- Get a strip near imaginary axis containing only finitely many QNF

Spectral gaps and expansions




$\Im s$
γ
$\Re s$

Incompleteness of the quasinormal spectrum

- Since the spectrum is discrete, might hope that any (smooth) solution can be expanded in QNM:

$$\psi(x,\tau) \overset{?}{=} \sum_{i=1}^{\infty} a_i u_i(x) e^{s_i \tau}, \quad a_i \in \mathbb{C}$$

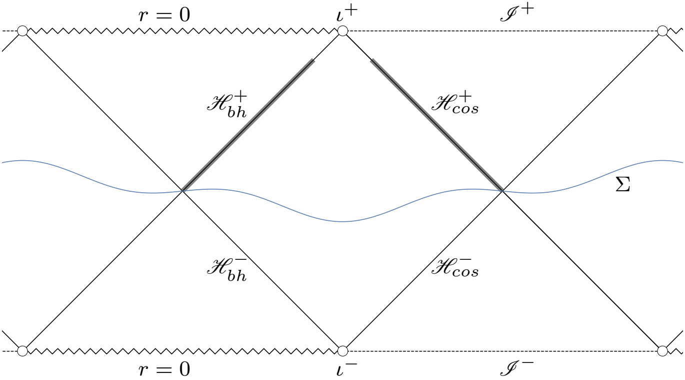
- This is *not* true.

Lemma

On a black hole background, there exist non-trivial solutions of the wave equation which vanish in a neighbourhood of ι^+.

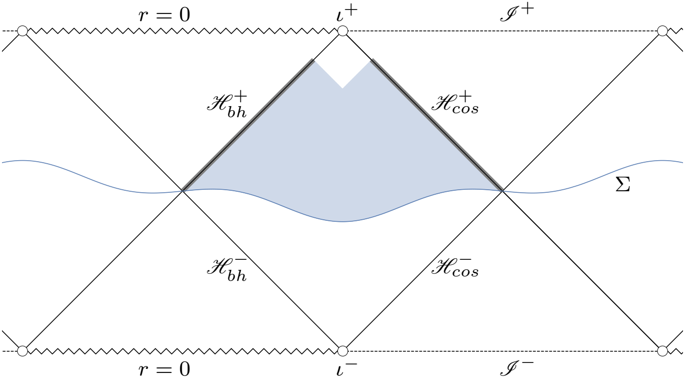
- Such a solution has a quasinormal expansion which is identically zero.
- Construction similar to backwards scattering construction of [Dafermos–Holzegel–Rodnianski '13; Dafermos–Rodnianski–Shlapentokh-Rothman '14]

Incompleteness of the quasinormal spectrum



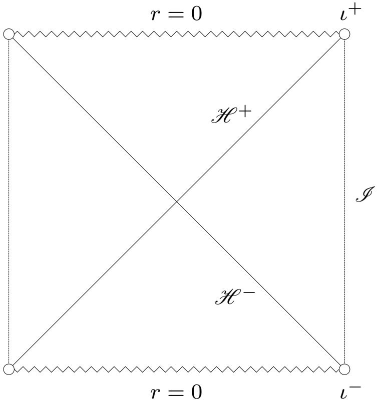
Specify data on $\mathscr{H}^+_{bh} \cup \mathscr{H}^+_{cos}$, vanishing near ι^+.
Solve wave equation backwards

Incompleteness of the quasinormal spectrum



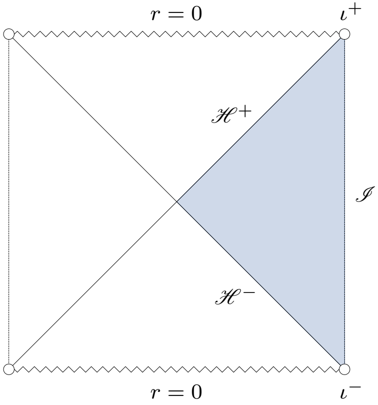
Solution induces data on Σ which evolve into a solution vanishing near ι^+.

Schwarzschild/Kerr–anti-de Sitter



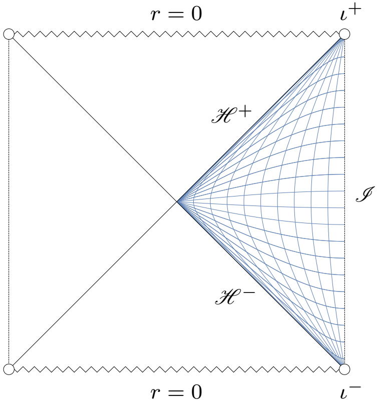
Schwarzschild–anti-de Sitter

Schwarzschild/Kerr–anti-de Sitter



Schwarzschild–anti-de Sitter

Schwarzschild/Kerr–anti-de Sitter



$$g = -\left(1 - \frac{2m}{r} + \frac{r^2}{l^2}\right) dt^2 + \frac{dr^2}{1 - \frac{2m}{r} + \frac{r^2}{l^2}} + r^2 d\Omega_{S^2}$$

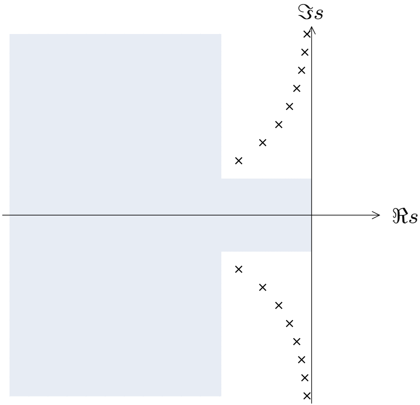
Schwarzschild/Kerr–anti-de Sitter

- Wave and Klein-Gordon equations on Schwarzschild/Kerr–Anti-de Sitter studied in [Holzegel '10; Holzegel–Smulevici '11; Holzegel–Warnick '14; Dold '15; Gannot '12, '14, '16]
- Stable trapping near $\mathscr{I}$ results in very slow decay:

$$\psi \sim \frac{1}{\log \tau}$$

- Slow decay manifests in quasinormal spectrum approaching the imaginary axis exponentially quickly

Schwarzschild/Kerr–anti-de Sitter

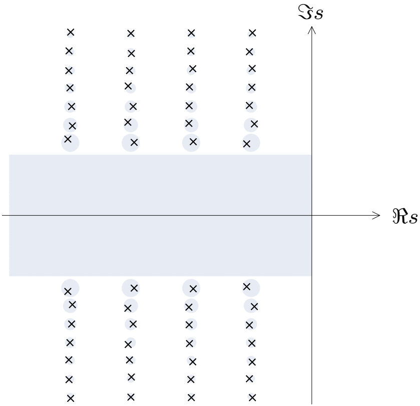


Schwarzschild–anti-de Sitter

Schwarzschild/Kerr–de Sitter

- Waves on Schwarzschild/Kerr–de Sitter studied in [Sá Baretto–Zworski '97; Dafermos–Rodnianski '07; Bony–Häffner '08; Vasy '13; Dyatlov '10, '11, '14; Hintz–Vasy '14, '15, '16]
- Trapping is normally hyperbolic
- Finitely many QNF in the region $-\nu < \Re s$ for some $\nu > 0$.
- For $|\Im s| \gg 1$, QNF located close to lattice obtained by WKB approximation.

Schwarzschild/Kerr–de Sitter



Schwarzschild–anti-de Sitter

Schwarzschild/Kerr–de Sitter stability

- For *linearised gravitational perturbations* about Schwarzschild–de Sitter,
  - all QNM with $\Re s \geq 0$, $\Im s \neq 0$ are pure gauge.
  - all QNM with $s = 0$ correspond to linearised Kerr solutions.
- Conclude that linear perturbations of Schwarzschild–de Sitter decay exponentially to a linearised Kerr solution.

Theorem (Hintz–Vasy '16)

The stationary region of a slowly rotating Kerr–de Sitter spacetime is nonlinearly stable as a solution of the vacuum Einstein equations $(\Lambda > 0)$.

Conclusion

- For black holes with $\Lambda \neq 0$, quasinormal modes can be defined in a mathematically satisfactory way as solutions of an eigenvalue problem
- Quasinormal modes do not form a complete basis
- Many numerical and heuristic results concerning specific black hole spacetimes can be put on a rigorous footing
- Understanding quasinormal modes is a key ingredient in recent stability results for Kerr–de Sitter.