# Hamilton Geometry
# -
# The geometry of spacetime from (modified) dispersion relations

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GR21
New York, Juli 2016



Leibniz Universität Hannover

1. Dispersion Relations and the Geometry of Spacetime

2. Hamiltonian Phase Space Geometry

3. Conclusion and Outlook

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# The dispersion relation and the geometry of spacetime in GR

The dispersion relation of a free point particle in general relativity

$$E^2 - p_\alpha p_\beta \delta^{\alpha\beta} = E^2 - \vec{p}^2 = m^2$$

- $m$ is the invariant mass parameter
- $E = g(\dot{\gamma},p)$ is the energy of to the particle
- $p_\alpha = g(e_\alpha,p)$ is the spatial momentum of to the particle

an observer on worldline $\gamma$ associates to the particle with 4-momentum $p$

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Covariant: The dispersion relation is a level set of a Hamilton function

$$H(x,p) = g^{ab}(x)p_a p_b = m^2$$

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Covariant: The dispersion relation is a level set of a Hamilton function

$$H(x,p) = g^{ab}(x) p_a p_b = m^2$$

The covariant dispersion relation demonstrates
- its intimidate relation to the geometry of spacetime, i.e. the metric $g$
- the geometry of spacetime is derived from
  second derivatives of $H$ w.r.t. the momenta p of particles
- particle worldlines are determined by Hamiltons equations of motion

# The dispersion relation and the geometry of spacetime in GR

## Planck scale modified dispersion relation of a free point particle

Doubly Special Relativity[Amelino-Camelia 2008], Relative locality [Amelino-Camelia, Freidel, Kowalski Gilkman, Smolin 2011]

$$E^2 - \vec{p}^2 = m^2 \quad \rightarrow \quad E^2 - \vec{p}^2 + \ell f(E, \vec{p}) + ... = m^2$$

- What is the underlying spacetime geometry?
- Higher orders in $E$ and $\vec{p}$ cannot yield metric spacetime geometry
- Relation between the particles 4-momentum p and E, $\vec{p}$?

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Idea: Start from covariant dispersion relation!

Covariant: The dispersion relation is a level set of a Hamilton function

$$H(x,p) = g^{ab}(x)p_a p_b + \ell G^{abc}(x)p_a p_b p_c + \ell^2 H^{abcd}(x) p_a p_b p_c p_d + ... = m^2$$

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The covariant dispersion relation encodes

- the geometry of spacetime, in the tensors $g$, $G$, $H$, …
- the geometry of spacetime is derived from second (and higher) derivatives of $H$ w.r.t. the momenta p of particles
- particle worldlines are determined by Hamiltons equations of motion

1. Dispersion Relations and the Geometry of Spacetime

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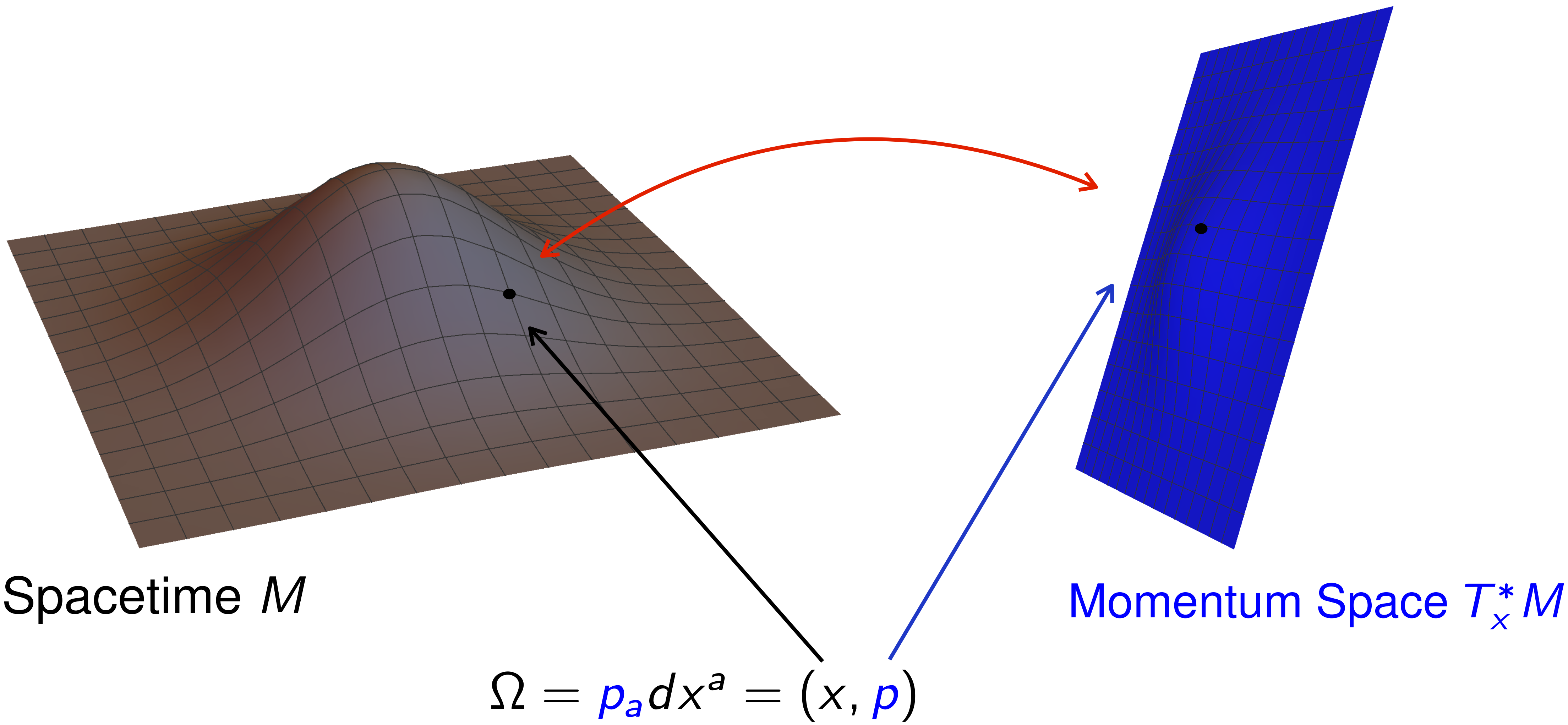
# Hamilton Geometry

**Hamilton phase space geometry**

- fundamental tupel ( T*M, H(x,p) ) cotangent bundle with Hamiltonian

**Metric spacetime geometry:**

- fundamental tupel ( M, g(x) ) spacetime with metric



# Hamilton Geometry

## Hamilton phase space geometry

- fundamental tupel ( T*M, H(x,p) )
  cotangent bundle with Hamiltonian
- the Cartan non-linear connection N determines the geometry of phase space

$$N_{ab} = \frac{1}{4}\left(\{g^H_{ab}, H\} + g^H_{ai}\partial_b \bar{\partial}^i H + g^H_{bi}\partial_a \bar{\partial}^i H\right)$$

$$g^{Hab} = \frac{1}{2}\frac{\partial}{\partial p_a}\frac{\partial}{\partial p_b}H = \frac{1}{2}\bar{\partial}^a\bar{\partial}^b H$$

unique, symmetric, torsion-free

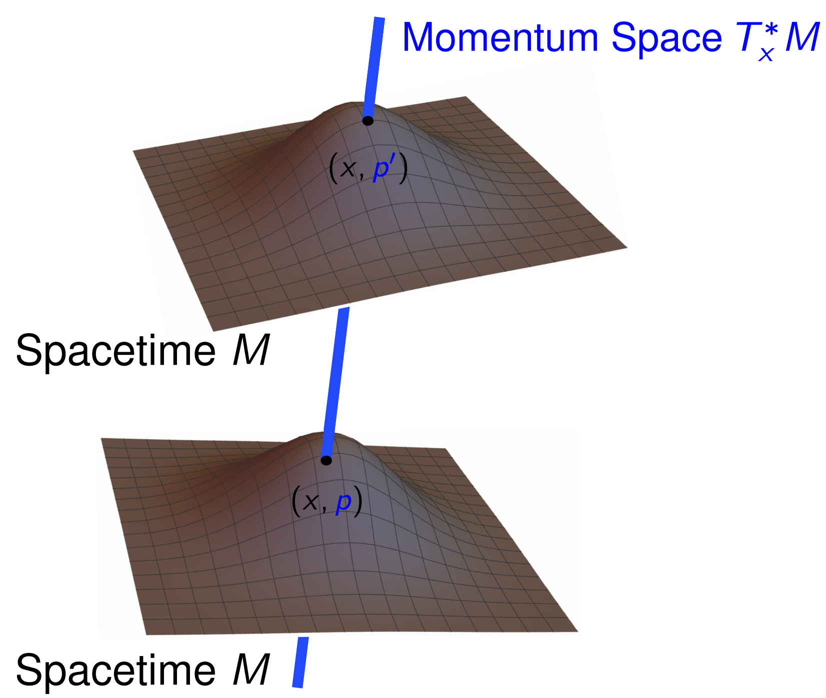
## Metric spacetime geometry:

- fundamental tupel ( M, g(x) )
  spacetime with metric
- the lifted Levi-Civita connection pΓ determines the geometry of phase space

$$p_a\Gamma^a{}_{bc} = \frac{1}{2}p_a g^{aq}\left(\partial_b g_{cq} + \partial_c g_{bq} - \partial_q g_{bc}\right)$$

$$g^{Hab} = g^{ab}$$

unique, symmetric, torsion-free



## Hamilton phase space geometry

- fundamental tupel ( T*M, H(x,p) ) cotangent bundle with Hamiltonian
- the Cartan non-linear connection N determines the geometry of phase space
- the spacetime connection and its curvature determine the geometry of spacetime

$$\Gamma^{\delta a}{}_{bc} = \frac{1}{2} g^{Haq} (\delta_b g^H_{cq} + \delta_c g^H_{bq} - \delta^H_q g^H_{bc})$$

$$\delta_a = \partial_a - N_{ab}(x,p)\bar{\partial}^b$$

$$R^a{}_{bcd}(x,p) = 2\delta_{[c}\Gamma^{\delta a}{}_{d]b} + 2\Gamma^{\delta a}{}_{q[c}\Gamma^{\delta q}{}_{d]b}$$

## Metric spacetime geometry:

- fundamental tupel ( M, g(x) ) spacetime with metric
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- the Levi-Civita connection and the Riemann curvature determine the geometry of spacetime

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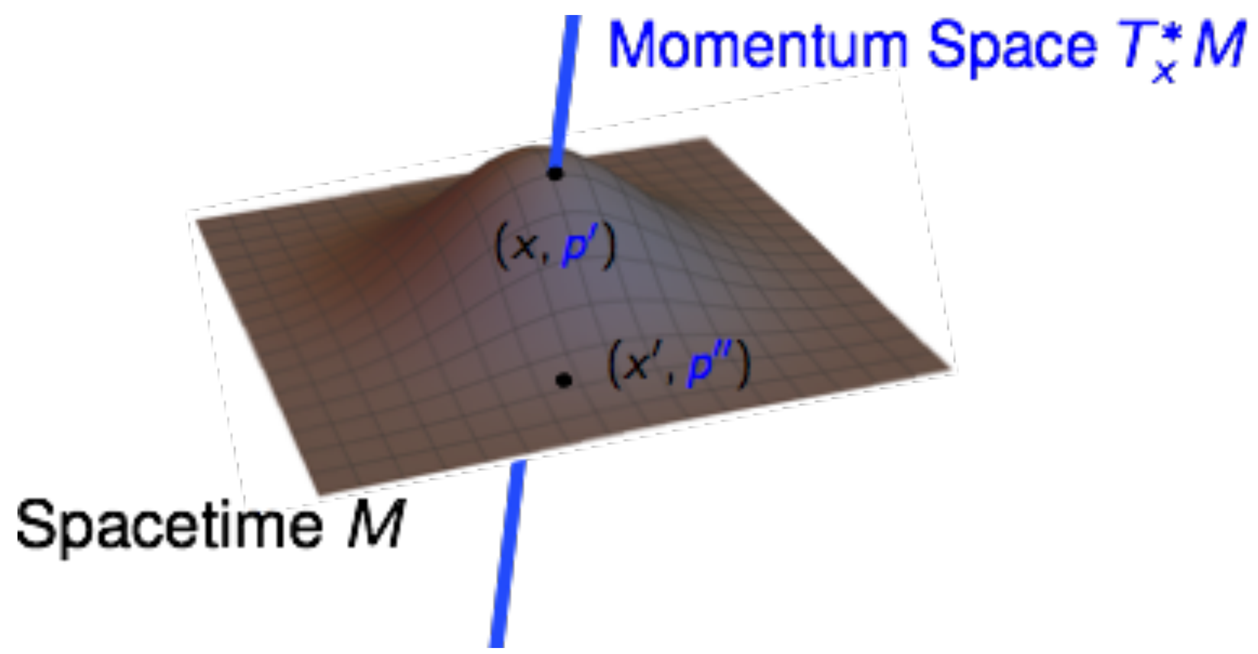

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- the spacetime connection and its curvature
  determine the geometry of spacetime
- the momentum space connection and its curvature determine the geometry of momentum space

$$C^{bc}{}_a = \tfrac{1}{2} g^H_{aq} \bar{\partial}^b g^{Hbq}$$

$$Q_a{}^{bcd}(x,p) = 2\bar{\partial}^{[c} C^{d]b}{}_a + 2 C^{q[c}{}_a C^{d]b}{}_q$$

## Metric spacetime geometry:

- fundamental tupel ( M, g(x) )
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- the lifted Levi-Civita connection pΓ
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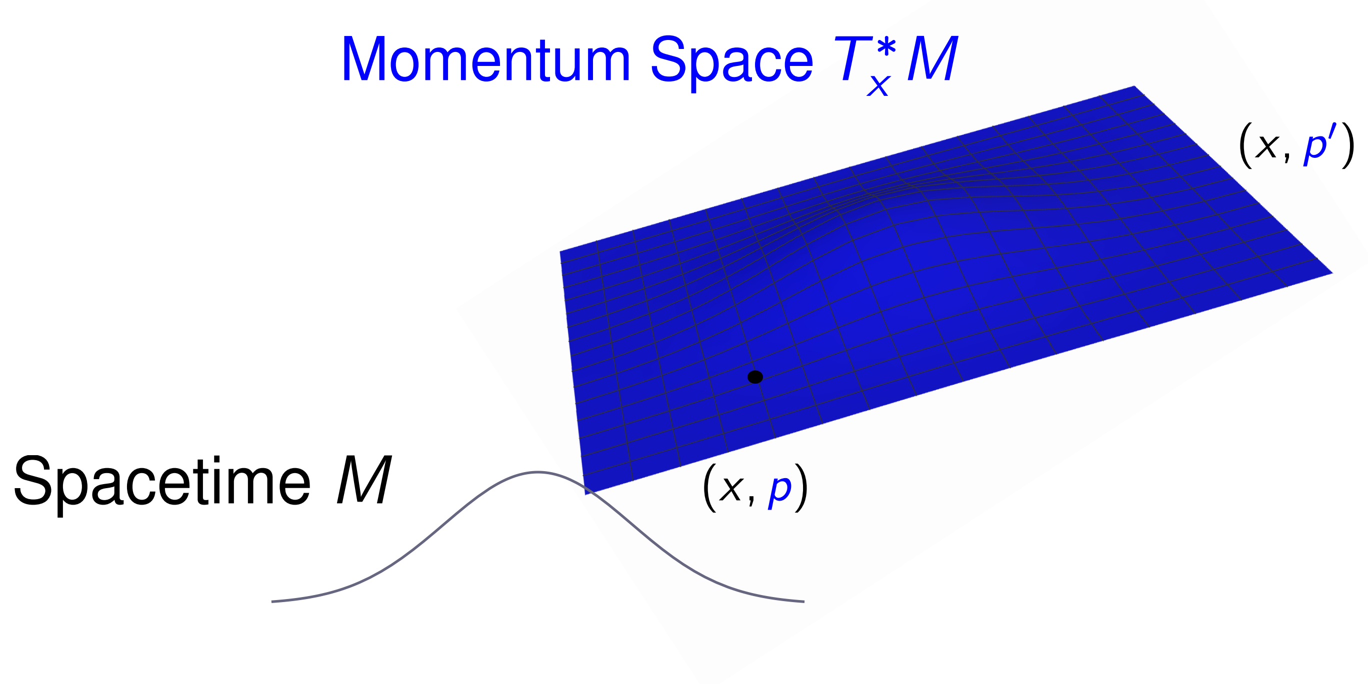
- the Levi-Civita connection and the Riemann curvature determine the geometry of spacetime
- the momentum space connection and its curvature vanish, momentum space is flat

$$C^{bc}{}_a = 0$$

$$Q_a{}^{bcd}(x,p) = 0$$



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### The geometry of a first order modified dispersion relation

[L. Barcaroli, L. Brunkhorst, G. Gubitosi, N. Loret, CP 2015]

$$H = g^{ab}(x)p_a p_b + \ell G^{abc}(x)p_a p_b p_c$$

$$N_{ab}(x,p) = -\Gamma^q{}_{ab}p_q + \ell\tfrac{3}{4}p_q p_r(\nabla_a G^{qr}{}_b + \nabla_b G^{qr}{}_a - g^{sq}\nabla_s G^r{}_{ab}) + \mathcal{O}(\ell^2)$$

$$\Gamma^{\delta a}{}_{bc}(x,p) = \Gamma^a{}_{bc} + \ell\tfrac{2}{3}p_q g^{ad}(\nabla_d G_{bc}{}^q - \nabla_b G_{cd}{}^q - \nabla_c G_{bd}{}^q) + \mathcal{O}(\ell^2)$$

$$C^{bc}{}_a(x,p) = \ell\tfrac{3}{2}G^{ab}{}_c + \mathcal{O}(\ell^2)\,.$$

# Hamilton Geometry

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The geometry of a first order modified dispersion relation

$$
\begin{aligned}
H &= g^{ab}(x)p_a p_b + \ell G^{abc}(x)p_a p_b p_c \\
N_{ab}(x,p) &= -\Gamma^q{}_{ab}p_q + \ell \tfrac{3}{4} p_q p_r (\nabla_a G^{qr}{}_b + \nabla_b G^{qr}{}_a - g^{sq}\nabla_s G^r{}_{ab}) + \mathcal{O}(\ell^2) \\
\Gamma^{\delta a}{}_{bc}(x,p) &= \Gamma^a{}_{bc} + \ell \tfrac{2}{3} p_q g^{ad}(\nabla_d G_{bc}{}^q - \nabla_b G_{cd}{}^q - \nabla_c G_{bd}{}^q) + \mathcal{O}(\ell^2) \\
C^{bc}{}_a(x,p) &= \ell \tfrac{3}{2} G^{ab}{}_c + \mathcal{O}(\ell^2)\,.
\end{aligned}
$$

The geometry of phase space is determined by three distinguished objects
- the Cartan non-linear connection
- the spacetime connection and its curvature
- the momentum space connection and its cuvature

all derived canonically from a Hamiltonian/disp. relation on phase space.

# Geometric interpretation of Hamilton equations of motion

Point particle follow solutions of Hamilton equations of motion

$$\dot{p}_a + \partial_a H = 0, \quad \dot{x}^a = \bar{\partial}^a H$$

In terms of the non-linear connection $N_{ab}$

$$\dot{p}_a + N_{ab}\bar{\partial}^b H = -[\partial_a - N_{ab}\bar{\partial}^b] H = -\delta_a H, \quad \dot{x}^a = \bar{\partial}^a H\,.$$

They can be compared to the autoparallels of the geometry

$$\dot{p}_a + N_{ab}\bar{\partial}^b H = 0.$$

In general point particles are not freely-falling but subject to a force term

$$\delta_a H$$

**Theorem:** [L. Barcaroli, L. Brunkhorst, G. Gubitosi, N. Loret, CP 2015]

For homogeneous Hamiltonians $H(x,s\,p) = s^r H(x,p)$
solutions to the Hamilton equations of motion are autoparallels:
the source term vanishes

$$\delta_a H = 0$$

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# Summary

Dispersion relations are level sets of Hamilton functions on phase space.

The geometry of phase space is determined by three distinguished objects
- the Cartan non-linear connection
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The spacetime and momentum space geometry is phase space dependent, i.e. depend on positions $x$ and momenta $p$.
An only $x$ dependent geometry of spacetime and only $p$ dependent geometry of momentum space is very special.

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An only *x* dependent geometry of spacetime and only *p* dependent geometry of momentum space is very special.

The geometric framework of Hamilton geometry allows a precise geometric comparison between general relativity and the geometry of spacetime induced by general (modified) dispersion relations.

# Outlook - Hamilton Geometry

Quantum gravity phenomenology:

- Geometric understanding of (modified) addition of momenta
- Geometric definition of observers, and transformations between them
- Study modified dispersion relation in homogeneous and isotropic, spherically symmetric or axially symmetric spacetime geometries

Applications to the analysis of PDEs:

- Dispersion relations are local representations of PDEs
- Geometric description of propagation of field modes.
  Example: Propagation of light in general linear electrodynamics, for example in media

$$H(x,p) = \mathcal{G}^{abcd}(x) p_a p_b p_c p_d$$

Dispersion relations are level sets of Hamilton functions:
They determine the geometry of spacetime and momentum space.

**Phys. Rev. D 92 (2015) 8, 084053**; arXiv:1507.00922:

**Hamilton geometry: Phase space geometry from modified dispersion relations**

Thank you for your attention