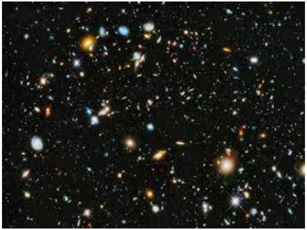
Flat and non-flat McVittie spacetimes: black holes versus the universe.



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Outline

- McVittie spacetimes: embedding of the Schwarzschild field in an FLRW universe
- Axioms and generalisation to non-flat FLRW backgrounds
- Global structure (radial null geodesics; $k = 0, \pm 1$)
- Bound particle and photon orbits ($k = 0$)
- Summary: apply some results from dynamical systems to the interpretation of an interesting (?) and important (??) solution of the Einstein equations

The metric: axiomatic approach

(C1) Spacetime (M, g) is spherically symmetric and is filled with a spherically symmetric, shear-free perfect fluid. Einstein's equation with a cosmological constant is satisfied.

(C2) The invariant Weyl scalar satisfies

$$\Psi_2 = -\frac{M}{r^3},$$

where M is a constant and r is the area-radius of the spacetime.

Useful invariants...

Lemma

If (C1) and (C2) hold, then there exist **invariant** *functions* A, f *of the cosmic time* t *such that*

$$\begin{aligned} \chi &:= g^{\alpha\beta} r_{,\alpha} r_{,\beta} = 1 - 2\frac{M}{r} + (A(t) - e^{2f(t)})r^2, \\ \psi &:= p^{\alpha\beta} r_{,\alpha} r_{,\beta} = 1 - 2\frac{M}{r} + A(t)r^2, \end{aligned}$$

where $p^{\alpha\beta} = g^{\alpha\beta} + u^\alpha u^\beta$ *is projection orthogonal to the fluid flow vector* u^α.

...in FLRW

Lemma

In an FLRW spacetime,

$$\begin{aligned} \chi &:= g^{\alpha\beta} r_{,\alpha} r_{,\beta} = 1 - (h^2 + k a^{-2}) r^2, \\ \psi &:= h^{\alpha\beta} r_{,\alpha} r_{,\beta} = 1 - \frac{k}{a^2} r^2, \end{aligned}$$

where $h^{\alpha\beta} = g^{\alpha\beta} + u^\alpha u^\beta$ *is projection orthogonal to the fluid flow vector* u^α, $a = a(t)$ *is the scale factor,* $h = \dot{a}/a$ *is the Hubble function and* k *is the curvature index.*

(C3) The functions A, f take the values found in an FLRW spacetime:

$$A(t) = \frac{k}{a^2}, \qquad e^{2f} = h^2.$$

Flat and non-flat

- **NB Use the area radius as a coordinate!**
- Bring in the remaining field equations...a further function of integration arises.
- In the flat case, $k = 0$, this is removed with a coordinate transformation: there is a unique solution which is McVittie's 1933 solution.
- In the non-flat case, we need an additional condition to uniquely specify the metric:

(C4) The pressure is homogeneous in the zero-mass limit: $\lim_{M \to 0} P(t, r) = P(t)$.

- This leads to a unique class of spacetimes depending on (i) a parameter M; (ii) a function $a(t)$ and (iii) an index $k = 0, \pm 1$.

Basic properties

Proposition

Given $D = (M, a(t), k)$, there is a unique spacetime $(\mathcal{M}, g)$ satisfying $(C1) - (C4)$.

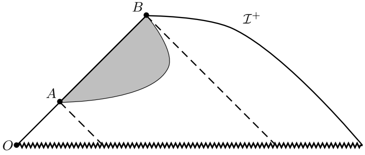
- *Explicit line element for $k = 0$; needs an elliptic integral $\phi = \phi(D)$ when $k = \pm 1$;*
- *fluid expansion is homogeneous: $\theta = \theta_0(t)$;*
- $8\pi\mu + \Lambda = 3(h^2 + ka^{-2})$,
  $8\pi P - \Lambda = -\phi^{-1}\frac{\partial}{\partial t}(h^2 + ka^{-2}) - 3(h^2 + ka^{-2})$;
- *$M = 0$ gives $FLRW_{k,a(t)}$;*
- *matter-free limit gives (i) Schwarzschild ($k \leq 0$); (ii) Schwarzschild-de Sitter (all k); (iii) Schwarzschild-Anti de Sitter ($k < 0$).*

Global structure

$$ds^2 \;=\; -\phi^2\left(h^{-2} - r^2\psi^{-1}\right)dt^2 - 2\phi r\psi^{-1}dtdr + \psi^{-1}dr^2 + r^2 d\Omega^2.$$

- In the $k = 0$ case, $\psi = 1 - \frac{2M}{r}$ forms a past space-like singular boundary ($P \to \infty$).
- In the $k = \pm 1$ cases, $\psi = 0$ is (essentially) a coordinate singularity (cf. $r = a(t)$ in the $k = +1$ FLRW spacetime). The Big Bang $a(t) = 0$ is the past boundary.
- For $\Lambda \geq 0$ and $h > 0$, future evolution of $k = -1$ is same as $k = 0$.

Class 1: $\lim_{t\to 0^+} a(t) = 0, h(t) > 0 \, \forall \, t > 0, \lim_{t\to +\infty} h(t) = \sqrt{3\Lambda} > 0$



- Penrose-Carter diagram for Class 1 McVittie.
- The singular boundary at $r = 2m$ forms the past boundary.
- There is an inner horizon at $r = r_-$, the inner horizon of the corresponding Schwarzschild-de Sitter spacetime.
- The boundaries at infinity are at infinite affine distance.
- Structure is universal for all Class 1 (cf. Kaloper et al (2010); Lake & Abdelqader (2011); Nolan (2014)).
- Kaloper et al conjectured, and Lake & Abdelqader demonstrated extendibility to Schwarzschild-de Sitter across the inner horizon.

Bound particle orbits

Proposition

Let $r = r_\circ$ be the radius of a stable particle orbit in Schwarzschild(-de Sitter) spacetime. The for initial values with $|H(t_0) - H_0|, |r(t_0) - r_\circ|, |\dot{r}(t_0)|$ sufficiently small, the particle orbit in ($\Lambda \geq 0, k = 0$) McVittie spacetime decays exponentially to a stable orbit (around $r = r_\circ$) of the Schwarzschild(-de Sitter) spacetime.

Sample sketch proof: particle orbits in $\Lambda = 0$

Lemma

Let $x = (H, r, \dot{r})$. There exists a C^2 function $w : \mathbb{R} \to \mathbb{R}$, with $w(0) = 0$, such that the geodesic equations read

$$\begin{aligned} \dot{H} &= w(H)u(x), && (1) \\ \dot{r} &= p, && (2) \\ \dot{p} &= -V'(r) + rH^2 + rf^{1/2}w(H)u(x)^2. && (3) \end{aligned}$$

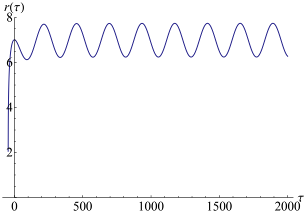
The C^2 function u is defined in a neighbourhood of $x_0 = (0, r_\circ, 0)$ and $\dot{t} = u(x)$.

- Key point: x_0 is a non-hyperbolic equilibrium point of the flow.

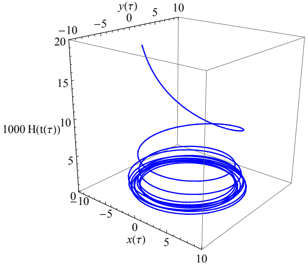
Schwarzschild dynamics $\to$ McVittie dynamics

- Recall that $H(t) \to 0$ as $t \to \infty$. $\{x : H = 0\}$ is an invariant manifold of the flow, corresponding to particle motion in Schwarzschild spacetime:
$$\ddot{r} + V'_{m,\ell}(r) = 0.$$
- Then
$$Q = \frac{1}{2}\dot{r}^2 + V_{m,\ell}(r) - V_{m,\ell}(r_\circ)$$
is a Lyapunov function for the flow.
- McVittie dynamics: Q is no longer conserved, but its evolution can be controlled by a Gronwall-type argument.
- Along the geodesic, $H(t(s))$ undergoes exponential decay, with overall exponential decay to an elliptic orbit of Schwarzschild spacetime.

Class 2 ($\Lambda = 0$): $M = 1, r_\circ = 7M, h(t) = \frac{2}{3}t^{-1}, \ell = 7/2$



- Plot of the area radius along the time-like geodesic.
- Note that $r \to 2M^+$ at finite proper time in the past.
- The geodesic is future complete.



- Spacetime representation of the geodesic; t and τ increase downwards.
- $H \to 0$ as $\tau \to \infty$, and the trajectory oscillates around $r = r_\circ = 7M$.

Conclusions

- The singular surface $\{r = 2M, t > 0\}$ forms a past boundary of the spacetime: **all** causal geodesics meet this surface at finite affine/proper time in the past. This cut-off is absent in the case $k = -1$: the Big Bang remains as the past boundary.
- Global structure results extend from $k = 0$ to $k = -1$ for all expanding, Big Bang FLRW backgrounds with $\Lambda \geq 0$.
- The central region can capture photons and particles, maintaining them in future complete, bound orbits.
- Class 1 and 2 McVittie spacetimes admit ISCO's, indicating possibility of formation of thin accretion disks. This is a black hole-like quality of these spacetimes.
- Model for an inhomogeneity in an FLRW background.
- See BC Nolan, CQG **31** 235008 (2014) and references therein.