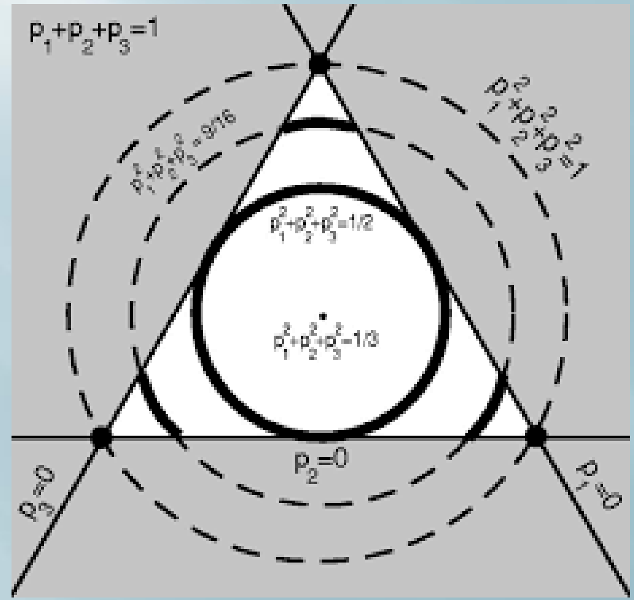
Einstein-Maxwell Solutions for the Propagation of Light in Anisotropic Kasner-Like Cosmological Epochs



Brett Bochner

Hofstra University, NY, USA

A Cosmological Truism (which *might actually* be true!) :

- Before the universe was Homogeneous & Isotropic,
  it was ***Inhomogeneous*** and ***Anisotropic***!

  (How did the Big Bang happen, anyway?)

- The focus of many cosmological studies is to ***get rid of*** the pre-Homogenized epoch as soon as possible *(we've gotten very good at this)*

- But given that the "initial" distribution of matter & energy is a great source of (aesthetic? philosophical? fine-tuning?) consternation, it is an interesting problem to study the propagation of relativistic mass-energy (e.g., light), during this Very-Early-Universe epoch

- A useful & well-known model of the anisotropic early universe is the ***Mixmaster model***, consisting largely of a varied series of Homogeneous-but-Anisotropically-Expanding "Kasner" Epochs

- Previous Kasner energy prop. studies often focused on approximations (light *ray* propagation, series expansions, neglected driving terms) or limited special cases (e.g., Kasner coefficients {2/3, 2/3, -1/3})

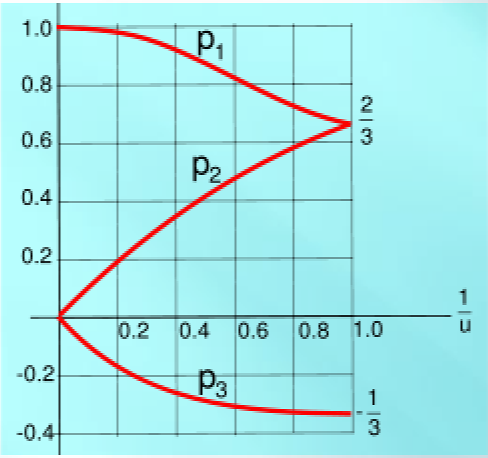
The Kasner metric:

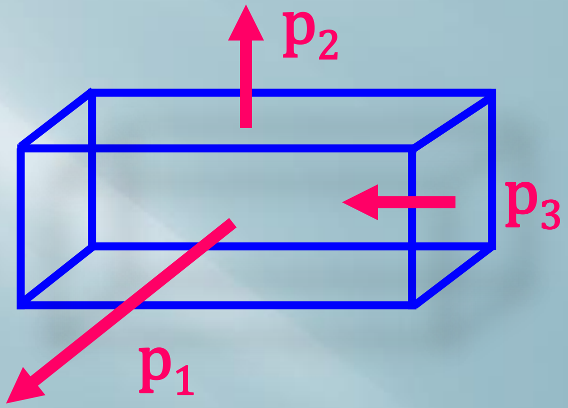
$$ds^2 = -dt^2 + (t^{2p_1})dx^2 + (t^{2p_2})dy^2 + (t^{2p_3})dz^2$$

With: $p_1 + p_2 + p_3 = {p_1}^2 + {p_2}^2 + {p_3}^2 = 1$,

$(\ldots\, for\ vacuum\ background{:}\ \ T^{\mu\nu} = 0)$,

And with: $p_1 > p_2 \geq p_3$, ➔ $p_3 \leq 0$ (*contracting*)





- But, there is *no particular reason* why we "must" enforce the Kasner conditions! Non-vacuum metrics (e.g., FRW) are perfectly acceptable (*whether or not* mass-energy influences the cosmological evolution), as long as reasonable Energy Conditions are satisfied.

➢ We will thus consider general $\{\boldsymbol{p_x}, \boldsymbol{p_y}, \boldsymbol{p_z}\}$, *without* prior restrictions.

Electromagnetism in Curved Spacetime:

Einstein-Maxwell eqn's:
(charge/current-free volumes)

$$\partial_\gamma F_{\alpha\beta} + \partial_\beta F_{\gamma\alpha} + \partial_\alpha F_{\beta\gamma} = 0$$

$$\partial_\alpha\{\sqrt{-\mathrm{Det}[g_{\mu\nu}]}\, F^{\alpha\beta}\} = 0$$

With E&B-Fields seen by Kasner (*Rest*-)Observer w/4-Velocity U^β :

$$E_\alpha{}^{Obs} = F_{\alpha\beta}\, U^\beta \;,\;\; B_\alpha{}^{Obs} = -\frac{1}{2}\epsilon_{\alpha\beta}{}^{\gamma\delta}\, F_{\gamma\delta}\, U^\beta \quad \{U^\beta = (1,0,0,0)\}$$

(where $\epsilon_{\alpha\beta}{}^{\gamma\delta}$ is the Levi-Civita anti-symmetric *tensor*, not *symbol*)

Hence (and w/cyclic permutations): $F_{tx} = -\sqrt{-g_{tt}}\, E_x{}^{Obs} = -E_x{}^{Obs}$,

$$F_{xy} = \sqrt{g_{xx}\, g_{yy}/g_{zz}}\, B_z{}^{Obs} = [\mathrm{t}\text{^}(p_x + p_y - p_z)]\, B_z{}^{Obs}$$

Also, to simplify the Divergence Eq's, define Re-Scaled Fields thus:

$$E_z \equiv (t^{p_x+p_y-p_z})\, E_z{}^{Obs} \;,$$

$$B_z \equiv (t^{p_x+p_y-p_z})\, B_z{}^{Obs} = F_{xy}$$

(and w/cyclic permutations over $\{x,y,z\}$)

Resulting Maxwell Equations for *general* $\{p_x, p_y, p_z\}$:

$$\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot\vec{B} = 0$$

$$\partial_t E_x = \{\,[t^{(-p_x\,-\,p_y\,+\,p_z)}]\,\partial_y B_z\} - \{\,[t^{(-p_x\,+\,p_y\,-\,p_z)}]\,\partial_z B_y\}$$

(➔ with Cyclic Perm. over $\{x,y,z\}$, and for $\{E_i \rightarrow\ B_i,\ B_i \rightarrow -E_i\}$)

Recalling the *usual trick* to get ***independent*** 2nd-Order Diff. Eq's. for the 6 Fields:

Minkowski (for a *Homogeneous* wave eq'n): $\partial_t\vec{E} = \vec{\nabla}\times\vec{B}\,,\ \partial_t\vec{B} = -\vec{\nabla}\times\vec{E}\,,$

$$\Rightarrow\ \vec{\nabla}\times\vec{\nabla}\times\vec{E} = \cancel{\vec{\nabla}(\vec{\nabla}\cdot\vec{E})}^{\,0} - \nabla^2\vec{E} = -\,\vec{\nabla}\times\overset{\leftrightarrow}{\partial_t}\vec{B} = -\,\partial_t\{\vec{\nabla}\times\vec{B}\} = -\,\partial_t^2\vec{E}\,,\ \boldsymbol{\partial_t^2\vec{E} - \nabla^2\vec{E} = 0}$$

BUT, for Kasner case, w/factors of t in the "curl" eq'ns: $\partial_t\{(t^N)\,\vec{B}\} \neq (t^N)\partial_t\vec{B}\,,$
a *Homogeneous* 2nd-Order wave eq'n is not obtained & we get (w/cyclic perm.):

$$\partial_t^2 E_x + \left(\frac{p_x}{t}\right)\partial_t E_x\ - \{(t^{-2p_x})\partial_x^2 + (t^{-2p_y})\partial_y^2 + (t^{-2p_z})\partial_z^2\}\,E_x$$

$$= \frac{(p_z-p_y)}{t}\,\{\,[t^{(-p_x\,-\,p_y\,+\,p_z)}]\,\partial_y B_z\} + \{\,[t^{(-p_x\,+\,p_y\,-\,p_z)}]\,\partial_z B_y\},$$

…so that the $\{E_i \leftrightarrow B_{j,k}\}$ fields are *not uncoupled* in the 2nd-Order eq'ns!

Uncoupling the *E*-, *B*-fields via the *Non-Homogeneous* "Driving" terms:

Using the "curl" eq'ns, we may write the 2nd-Order Diff Eq'n for a given field (e.g., E_x) in 3 different ways… (and defining: $\{t\nabla^2\}E_i \equiv \{(t^{-2p_x})\partial_x^2 + (t^{-2p_y})\partial_y^2 + (t^{-2p_z})\partial_z^2\}\, E_i$):

$$\partial_t^2 E_x + \left(\frac{p_x}{t}\right) \partial_t E_x - \{t\nabla^2\}E_x = \frac{(p_z - p_y)}{t} \{\, [t^{(-p_x - p_y + p_z)}]\, \partial_y B_z\} + \{\, [t^{(-p_x + p_y - p_z)}]\, \partial_z B_y\}$$

or:

$$\partial_t^2 E_x + \left[\frac{p_x + (p_z - p_y)}{t}\right] \partial_t E_x - \{t\nabla^2\}E_x = 2\, \frac{(p_z - p_y)}{t} \{\, [t^{(-p_x - p_y + p_z)}]\, \partial_y B_z\}\}$$

or:

$$\partial_t^2 E_x + \left[\frac{p_x - (p_z - p_y)}{t}\right] \partial_t E_x - \{t\nabla^2\}E_x = 2\, \frac{(p_z - p_y)}{t} \{\{\, [t^{(-p_x + p_y - p_z)}]\, \partial_z B_y\}$$

Doing this appropriately for all 6 fields, the 6 $\{E_i \leftrightarrow B_{j,k}\}$ coupled eq'ns can be broken into 3 sets of $\{E_i \leftrightarrow B_j\}$ pairwise-coupled eq'ns… e.g., $\{E_x \leftrightarrow B_y\}$, $\{E_y \leftrightarrow B_z\}$, & $\{E_z \leftrightarrow B_x\}$;

Then, for all 6 fields $F_i \equiv \{E_i, B_i\}$, w/spatial dependence: $\cos/\sin(k_x x + k_y y + k_z z)$,

$$-\{t\nabla^2\}F_i \equiv \left[(t^{-2p_x})k_x^2 + (t^{-2p_y})k_y^2 + (t^{-2p_z})k_z^2\right] F_i \;,$$

and we can thus take extra spatial derivatives, replace w/k's, and plug the pairwise-coupled into one another, to remove the coupling… e.g.:

$\partial_z\, \partial_t^2 E_x \sim -k_z^2\, B_y$ ➔ *(eq'n for)* $\partial_t^2 B_y$ ➔ *(more spatial deriv's)* ➔ *(<u>uncoupled</u> eq'n in* $\boldsymbol{\partial_t^4 E_x}$ *!)*

Fully-Uncoupled **4th-Order** Diff Eq'n (for, e.g.,) $F_x \equiv \{E_x, B_x\}$:

$$\partial_t^4 F_x$$

$$+ \left[\frac{2(1+2p_x)}{t}\right] \partial_t^3 F_x$$

$$+ \left\{ 2\left[(t^{-2p_x})k_x^2 + (t^{-2p_y})k_y^2 + (t^{-2p_z})k_z^2\right] + \frac{1}{t^2}\left[2p_x + 5p_x^2 - (p_y - p_z)^2\right] \right\} \partial_t^2 F_x$$

$$+ \left\{ \frac{2}{t}\left[(t^{-2p_x})k_x^2 + \left[(t^{-2p_y})k_y^2(1+2p_x-2p_y)\right] + \left[(t^{-2p_z})k_z^2(1+2p_x-2p_z)\right]\right] + \frac{1}{t^3}\left((2p_x - 1)\left[p_x^2 - (p_y - p_z)^2\right]\right) \right\} \partial_t F_x$$

$$+ \left\{ \left[(t^{-2p_x})k_x^2 + (t^{-2p_y})k_y^2 + (t^{-2p_z})k_z^2\right]^2 + \frac{2}{t^2}\left[(t^{-2p_y})\, k_y^2(p_x - p_y)(1 + p_x + p_z - 3p_y) + (t^{-2p_z})k_z^2(p_x - p_z)(1 + p_x + p_y - 3p_z) \right]\right\} F_x$$

$$= 0$$

(& equiv. 4th-Order Diff Eq'ns for $F_y \equiv \{E_y, B_y\}$ & $F_z \equiv \{E_z, B_z\}$ via *cyclic permutations*)

➔ Each field has **4** sol'ns, *not* 2! (…*except* for Kasner cases & polarizations where the 2nd-Order Driving Terms on the R.H.S. are ***zero***.)

Exact Soln's for an interesting (though conformally flat & vacuum Kasner)

Special Case, good for *general* $\vec{k} = (k_x, k_y, k_z)$: $\{\boldsymbol{p_x}, \boldsymbol{p_y}, \boldsymbol{p_z}\} = \{\boldsymbol{p_1}, \boldsymbol{p_2}, \boldsymbol{p_3}\} = \{\mathbf{1}, \mathbf{0}, \mathbf{0}\}$

In *this* case, the 2nd-Order eq'ns for $F_x \equiv \{E_x, B_x\}$ are Homogeneous:

$$\partial_t^2 F_x + \frac{1}{t}\partial_t F_x + \left[\frac{k_x^2}{t^2} + \left(k_y^2 + k_z^2\right)\right] F_x = 0\,,$$ thus possessing the **2** soln's:

$$F_x \propto J_{\pm i k_x}\left[\sqrt{k_y^2 + k_z^2}\; t\right]$$ (…taking the two ***real*** combinations.)

For the 4th-Order eq'ns for $F_y \equiv \{E_y, B_y\}$ & $F_z \equiv \{E_z, B_z\}$, the **4** sol'ns (good for each) can be *guessed* from Bessel recursion relations:

$$F_{y,z} \propto t\left\{J_{(\pm i k_x - 1)}\left[\sqrt{k_y^2 + k_z^2}\; t\right] + J_{(\pm i k_x + 1)}\left[\sqrt{k_y^2 + k_z^2}\; t\right]\right\} \propto J_{\pm i k_x}\left[\sqrt{k_y^2 + k_z^2}\; t\right]$$

(…are 2 good soln's, and …)

$$F_{y,z} \propto t\left\{J_{(\pm i k_x - 1)}\left[\sqrt{k_y^2 + k_z^2}\; t\right] - J_{(\pm i k_x + 1)}\left[\sqrt{k_y^2 + k_z^2}\; t\right]\right\} \propto t\left\{\partial_t\, J_{\pm i k_x}\left[\sqrt{k_y^2 + k_z^2}\; t\right]\right\}$$

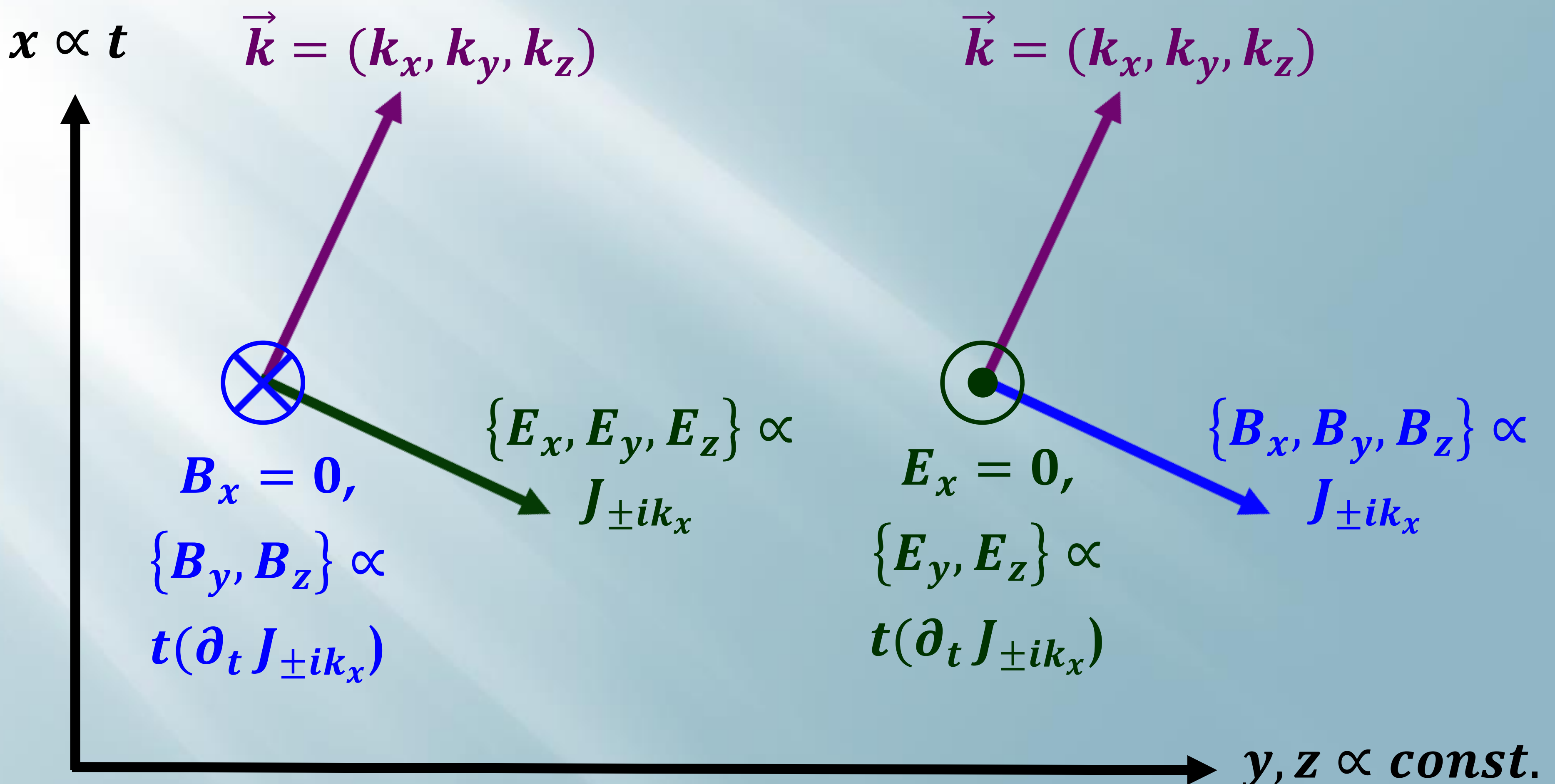
(…happen to be the *other* 2 good sol'ns.)

Finally, considering Polarization states, these soln's can be *illustrated* as follows…

Matching these 4th-Order Soln's via the Maxwell Eq'ns for this $\vec{p} = (1,0,0)$ Kasner case, the fields can be written as superpositions of 2 diff. Polarizations:

Nonzero $\boldsymbol{E_x}$ drives $\boldsymbol{B_y}$ & $\boldsymbol{B_z}$:

Nonzero $\boldsymbol{B_x}$ drives $\boldsymbol{E_y}$ & $\boldsymbol{E_z}$:



The Difficulty in Solving cases w/ *general* Kasner **p-Values**…

We can often "guess" the 4 soln's for the "driven" fields w/Nonhomogeneous 2^{nd}-Order eq'ns, from the 2 soln's for the Homogeneous eq'ns (R.H.S. always zero for *some* polarization)… *but **only** if the 2^{nd}-Order eq'ns are solvable* :

$$\partial_t^2 F_i + \frac{N(p's)}{t}\partial_t F_i + \left[\frac{k_x^2}{t^{2p_x}} + \frac{k_y^2}{t^{2p_y}} + \frac{k_z^2}{t^{2p_z}}\right] F_i = 0$$

➔ This is not typically solvable *(to my knowledge)* in terms of the usual Bessel Functions, for general values of (p_x, p_y, p_z) !

So, "Solvable Cases":

❖ "Early-Time" or "Late-Time" Approximations (let $p_x > \{p_y, p_z\}$),
$[(t^{-2p_x})k_x^2] \gg \left[\left(t^{-2p_y}\right)k_y^2 + (t^{-2p_z})k_z^2\right]$, or, $\left[(t^{-2p_x})k_x^2 + \left(t^{-2p_y}\right)k_y^2\right] \ll [(t^{-2p_z})k_z^2]$

➢ Unique & Consistent 4^{th}-Order eq'ns cannot be produced, unless *two* of $\{k_x, k_y, k_z\}$ are ***zero***… ➔ 1-Dimensional Propagation! (…all solved here as *Special Cases*…)

❖ 2-Dimensional Propagation – e.g., $\vec{k} = (k_x, k_y, 0)$ – but need $p_x = 1$ *and/or* $p_y = 1$

❖ {…Soln's for *general* (p_x, p_y, p_z) still being sought!}

Solvable cases w/2-Dimensional* Propagation, with: $\vec{\boldsymbol{k}} = (\boldsymbol{k_x}, \boldsymbol{k_y}, \boldsymbol{0})$, $\boldsymbol{p_x} \geq \boldsymbol{p_y}$

(* Or, *3-Dimensional Propagation* with Cylindrical Symmetry…

if $p_z = p_y$, k_y ➔ $\sqrt{k_y^2 + k_z^2}$; if $p_z = p_x$, k_x ➔ $\sqrt{k_x^2 + k_z^2}$)

(I) $\boldsymbol{p_x = 1}$, $\boldsymbol{p_y < 1}$ *(No Horizon Prob. in x-dir.)*: Soln's ~ $\{(t^{-\Delta p})\, J_{\pm ORD}[ARG]\}$, with:

$$\Delta p = \pm\frac{1}{2}(p_y - p_z) \quad , \quad ORD = \frac{\sqrt{(\Delta p)^2 - k_x^2}}{(1-p_y)} \quad , \quad \mathrm{ARG} = \frac{k_y}{(1-p_y)}\ t^{(1-p_y)}$$

(N.B.: ORD only complex – typically oscillatory – for $\{k_x > |\Delta p|\}$)

(II) $\boldsymbol{p_x > 1}$, $\boldsymbol{p_y = 1}$ *(No Horizon Prob. in $\{x, y\}$-dir's)*: Soln's ~ $\{(t^{-\Delta p})\, J_{\pm ORD}[ARG]\}$, with:

$$\Delta p = \frac{1}{2}[p_x - 1 \pm (1 - p_z)] \quad , \quad ORD = \frac{\sqrt{(\Delta p)^2 - k_y^2}}{(1-p_x)} \quad , \quad \mathrm{ARG} = \frac{k_x}{(1-p_x)}\ t^{(1-p_x)}$$

*(N.B.: ORD only complex for $\{k_y > |\Delta p|\}$; **And, $\boldsymbol{ARG \rightarrow -\infty}$, as $\boldsymbol{t \rightarrow 0}$**)*

(III) $\boldsymbol{p_x = p_y = 1}$, with: $k \equiv \sqrt{k_x^2 + k_y^2}$, $\Delta p = \frac{1}{2}(1 - p_z)$,

Soln's ~ $\{[t^{\mp\Delta p}] \cos/\sin[\sqrt{k^2 - (\Delta p)^2}\, \ln t]\}$ *(N.B.: Only oscillatory for $\{k > |\Delta p|\}$)*