Post Newtonian Approximation of Black Hole Entropy

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Work in progress...

Introduction

Motivation

In this study we are considering non-relativistic bosons around a spherical object. Therefore it can naturally adapt to astromical objects surrounding with clouds, such as forming planets, black holes, among many others.

Steps to follow

We will take fat and slow particles (non-relativistic) around a spherical object (curved space-time). First, we will examine the simplest case: ultrastatic metric.Then examine a generic case: static metric. Afterwards we will rescale everthing back to the simplest case by a γ metric. Eventually, we turn the object into a black hole and look for its thermodynamical features.

Non-Relativistic Limit in Ultrastatic Background

We are working with a non-relativistic (NR) system on a Riemanian manifold with an ultrastatic metric

$$ds^2 = -dt^2 + h_{ij}dx^i dx^j \tag{1}$$

Klein-Gordon equation (KGE) is given by

$$-\partial_0^2\psi - \Delta_h\psi + m^2\psi + U\psi = 0. \tag{2}$$

Through curved effects the potential U will include the scalar curvature term ξR. Via the conformal transformation the Laplacian becomes

$$\Delta_h = \frac{1}{\sqrt{h}}\partial_i\sqrt{h}h^{ij}\partial_j \tag{3}$$

For NR particles the large m limit is taken and expansion parameter x will be defined as $(Tm)^{-1}$ since it is small.

Bose-Einstein Distribution in the Flat Space-Time

The distribution function of bosons in the flat space is

$$n_i = \frac{1}{e^{\beta(\epsilon_i - \mu)} - 1} \qquad \beta = \frac{1}{k_B T} \tag{4}$$

Therefore the free energy can be written as below even in a flat space-time

$$\mathcal{F} = \frac{1}{\beta} \sum_{\sigma} \log(1 - e^{-\beta(\epsilon_\sigma - \mu)}) \tag{5}$$

$$= - \sum_{k=1}^{\infty} \frac{1}{k\beta} e^{k\beta\mu} \sum_{\sigma} e^{-k\beta\epsilon_\sigma} \tag{6}$$

Helmholtz Free Energy

The free energy is

$$\mathcal{F} = \frac{1}{\beta} \sum_\sigma \log\left(1 - e^{-\beta(\epsilon_\sigma - \mu)}\right), \tag{7}$$

which upon expanding the logarithm can also be written as

$$\mathcal{F} = -\sum_{k=1}^{\infty} \frac{1}{k\beta} e^{k\beta\mu} \sum_\sigma e^{-k\beta\epsilon_\sigma}. \tag{8}$$

We compansate the order difference between ϵ_σ and ϵ_σ^2 with the identity

$$e^{-b\sqrt{x}} = \frac{b}{2\sqrt{\pi}} \int_0^\infty \frac{du}{u^{3/2}}\, e^{-\frac{b^2}{4u}}\, e^{-ux} \tag{9}$$

$$\mathcal{F} = -\frac{c}{2\sqrt{\pi}} \sum_{k=1}^{\infty} e^{k\beta\mu} \int_0^\infty \frac{du}{u^{3/2}}\, e^{-\frac{(k\beta c)^2}{4u}} \sum_\sigma e^{-u\epsilon_\sigma^2} \tag{10}$$

The Saddle Point Method

Now we derive the NR limit of this expression by taking the large m limit by the saddle point method. Note that $\beta^{-1}m$ is kept fixed in taking this limit.

$$\mathcal{F} = -\frac{c}{2\sqrt{\pi}} \sum_{k=1}^{\infty} e^{k\beta\mu} \int_0^{\infty} \frac{du}{u^{3/2}}\, e^{-(mc)^2\left(\frac{(k\beta m^{-1})^2}{4u}+u\right)}\, Tre^{-u(-\Delta_h+U)} \tag{11}$$

Now the saddle point $\overline{u}$ is given by the solution of

$$\frac{d}{du}\left[\frac{(k\beta m^{-1})^2}{4u}+u\right] = 0 \quad as \quad \overline{u} = \frac{k\beta}{2m}. \tag{12}$$

Harmonic Sum

Putting everything together we get

$$\mathcal{F} \;=\; -\sum_{k=1}^{\infty} \frac{1}{k\beta}\, e^{k\beta\mu}\, Tr\, e^{-k\beta\left(\frac{1}{2m}(-\Delta_h+U)+mc^2\right)} \tag{13}$$

Let us define $x = \frac{\beta}{2m}$. $\mathcal{F}$ is in the form of a harmonic sum

$$\mathcal{F} = \sum_{k=1}^{\infty} f(kx) \tag{14}$$

where

$$f(x) = -\frac{1}{2mx} e^{x(2m\mu)}\, Tr\, e^{-x\left(-\Delta_h+2mU+2(mc)^2\right)}. \tag{15}$$

Mellin Transformation

The Mellin transformation is

$$\tilde{F}(s) = \int_0^\infty u^{s-1} f(u)\, du \quad (16)$$

$$f(u) = e^{\mu u} Tr' e^{-Hu} \quad (17)$$

By using the Mellin transformation, we find asymptotic behaviour of the free energy from the poles of the below dictionary.

$$(\mathcal{M}F)(s) \asymp \sum_{wk} \frac{R(w,k)}{(s-w)^{k+1}} \quad (18)$$

$$\updownarrow$$

$$F(s) = \sum_{wk} R(w,k) \frac{(-1)^k}{k!} s^{-w} (\log s)^k \quad (19)$$

Extensions and Expansions

The meromorphic extension of the harmonic sum is

$$\widetilde{\mathcal{F}}(s) = \zeta(s)\widetilde{f}(s) \tag{20}$$

where

$$\widetilde{f}(s) = -\frac{1}{2m(4\pi)^{3/2}}\sum_{j=0}^{\infty} a_{j/2}(2(m^2c^2 - m\mu))^{-s-\frac{j-5}{2}}\Gamma\left(s + \frac{j-5}{2}\right). \tag{21}$$

The singular expansion of Γ and Laurent series expansion of the ζ are used to expose the poles of the function $\mathcal{F}$

$$\Gamma(x) \asymp \sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\frac{1}{x+l}, \qquad \zeta(s) = \frac{1}{s-1} + \gamma + O(s-1) \tag{22}$$

Bose Einstein Condensates in Curved Space Time

Since our particles are confined in a spherical box, heat kernel expansion is being used to avoid the expression of the density of states in the curved space.

$$Tre^{\Delta u} = \frac{1}{(4\pi u)^{3/2}}\left[a_0 + a_{1/2}u^{1/2} + a_1 u + \ldots\right] \quad (23)$$

where the heat kernel coefficients can be found according to the metric.

$$\begin{aligned} a_0 &\sim V, \\ a_{1/2} &\sim S \end{aligned} \quad (24)$$

Free Energy and Entropy

Eventually for the ultrastatic case we find

$$\begin{aligned}\mathcal{F} &= -\frac{\zeta(5/2)}{(4\pi)^{3/2}2m}a_0\,(2mT)^{5/2}-\frac{\zeta(2)}{(4\pi)^{3/2}2m}a_{1/2}\,(2mT)^2 \\ &\quad -\frac{\zeta(3/2)}{(4\pi)^{3/2}2m}[a_1-2((mc)^2-m\mu)a_0]\,(2mT)^{3/2}+\ldots\end{aligned} \quad (25)$$

therefore

$$\begin{aligned}\mathcal{S} &= +\frac{5}{2}\frac{\zeta(5/2)}{(4\pi)^{3/2}}a_0\,(2mT)^{3/2}+2\frac{\zeta(2)}{(4\pi)^{3/2}}a_{1/2}\,(2mT) \\ &\quad +\frac{3}{2}\frac{\zeta(3/2)}{(4\pi)^{3/2}}[a_1-2((mc)^2-m\mu)a_0]\,(2mT)^{1/2}+\ldots\end{aligned} \quad (26)$$

Static Background Case

The new metric is

$$ds^2 = -F(r)dt^2 + h_{ij}dx^i dx^j. \tag{27}$$

$F(r)$ is the time lapse function. KGE is written as

$$-\partial_0^2\Psi + \frac{F}{\sqrt{-g}}\partial_i\sqrt{-g}h^{ij}\partial_j\Psi - m^2F\Psi - \xi R\Psi = 0 \tag{28}$$

Through taking the square root and expanding in m^{-2} the NR Hamiltonian is found as

$$H'_{NR} = -\frac{1}{2m}F^{1/4}\frac{1}{\sqrt{-g}}\partial_i\sqrt{-g}h^{ij}\partial_i F^{1/4} + mF^{1/2}. \tag{29}$$

The Optical Metric

The heat kernel expansion requires a pure Laplacian plus a potential. By introducing the optical metric, we solve this problem by getting rid of the lapse function.

$$\gamma_{ij} = F^{-1} h_{ij}, \quad \Psi_\gamma = F^{d-1} \Psi \tag{30}$$

We get

$$H_\gamma = c^2[-\Delta_\gamma + (mc)^2 + U], \tag{31}$$

with

$$\Delta_\gamma = \frac{F^3}{\sqrt{-h}} \partial_i F^{-1} \sqrt{-h} h^{ij} \partial_j \tag{32}$$

A replica of the static case is obtained.

Including the Physical Surrounding

Eventually $\mathcal{F}$ and $\mathcal{S}$ turn into

$$\mathcal{F} = \mathcal{F}(M, T, \epsilon), \quad \textit{and} \quad \mathcal{S} = \mathcal{S}(M, T, \epsilon) \tag{33}$$

Therefore with the proper limits of a black hole, Hawking temperature, and brickwall cutoff

$$M^2 = \frac{A_H}{16\pi}, \quad T \to T_H = \frac{1}{8\pi M} \quad \textit{and} \quad \epsilon \to \frac{\delta^2}{8M} \tag{34}$$

Entropy has a general expression for different metrics: Schwarzschild, Dilaton, and Reissner–Nordström.

$$\mathcal{S} = \frac{A_H}{\delta^2} f_1(m^2 A_H) + \ln\left(\frac{2{A_H}^{1/2}}{\pi^{1/2}\delta^2}\right) f_2(m^2 A_H) \tag{35}$$

Results with Surface Gravity

When the proper limits with the surface gravity is being used

$$A_H = \frac{\pi}{K^2}, \quad T \to T_H = \frac{K}{2\pi} \quad and \quad \epsilon \to \frac{K^2\delta^2}{2} \tag{36}$$

Again entropy is seen to have a universal expression

$$\mathcal{S} = \left[\frac{5\zeta(5/2)}{2^8\pi^{9/2}} m^{3/2} K^{-3/2} A_H - \frac{\zeta(2)}{2^6\pi^{7/2}} m K^{-1} A_H - \frac{3\zeta(3/2)}{2^8\pi^{7/2}} m^{5/2} K^{-5/2} A_H + \frac{\zeta(3/2)}{2^6\pi^{7/2}} m^{1/2} K^{-1/2} A_H\right] \frac{1}{\delta^2} \tag{37}$$

Thank you!!