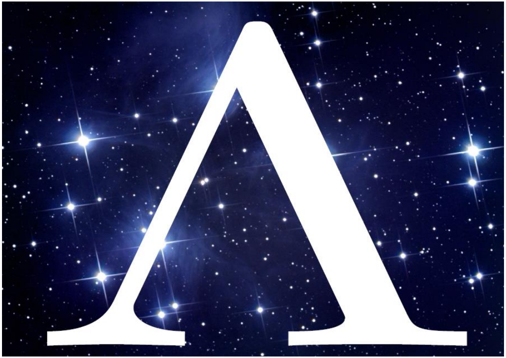
Asymptotics with $\Lambda>0$



Béatrice Bonga, Penn State

Work in collaboration with A. Asthekar and A. Kesavan [PRD 2015 + PRL 2016]

Gravitational waves with $\Lambda = 0$

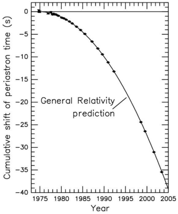
1916
Einstein predicts gravitational waves (GWs)

1960s
Bondi et al:
"They are real!"

Hot debate whether GWs are real

1974
Observation by Hulse and Taylor confirms this

Today



Gravitational waves with $\Lambda > 0$

Open issues

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- No ‘peeling’ theorem
- No positivity of energy
- No positive and negative frequency decomposition needed for Hilbert spaces in quantum theory

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- ... and no quadrupole formula!

Outline

1. Steps calculation in Minkowski space-time
2. Steps calculation in de Sitter space-time
3. Result

Linearization

One-parameter family of metrics

$$g_{\mu\nu}(\epsilon) = g_{\mu\nu}(0) + \epsilon\, h_{\mu\nu} + \dots$$

- For Minkowski: $g_{\mu\nu}(0) = \eta_{\mu\nu}$
- For de Sitter: $g_{\mu\nu}(0) = a^2(\eta)\mathring{g}_{\mu\nu}$ with $a(\eta) = -\frac{1}{\sqrt{\Lambda/3}\eta}$

Main ingredients

3 main ingredients

1. gauge fixed gravitational wave $\bar{h}_{\mu\nu}$

$$\Box\bar{h}_{\mu\nu} = 16\pi G T_{\mu\nu} \Longrightarrow \bar{h}_{ij} = \frac{4G}{r}\int \mathrm{d}^3x'\, T_{ij}(t-r, x')$$

2. energy & power

$$H_t = \frac{1}{32\pi G}\int \mathrm{d}^3x \left(\dot{h}_{ij}\dot{h}^{ij} - h_{ij}\ddot{h}^{ij}\right)$$

$$P_t := \mathcal{L}_t H_t = \frac{\partial}{\partial t} H_t$$

3. quadrupole moment & conservation of stress-energy tensor

$$Q_{ij} := \int \mathrm{d}^3x\, \rho\, x_i x_j$$

$$\int \mathrm{d}^3x T_{ij} = \frac{1}{2}\int \mathrm{d}^3x\, \ddot{T}_{00} x_i x_j = \frac{1}{2}\ddot{Q}_{ij}$$

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Energy loss in Minkowski

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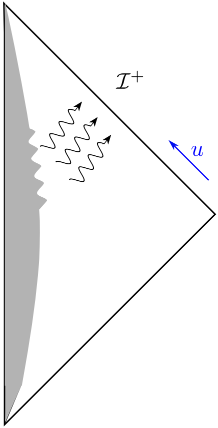
Energy and power

$$H_t = \frac{G}{8\pi} \int \mathrm{d}u \int \mathrm{d}\Sigma_2 \, \dddot{Q}_{ij}^{TT}(u) \dddot{Q}^{ij}_{TT}(u)$$

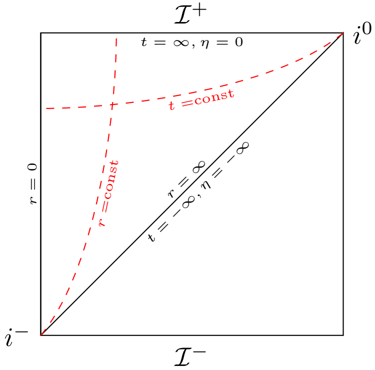
$$P_t = \frac{G}{8\pi} \int \mathrm{d}\Sigma_2 \, \dddot{Q}_{ij}^{TT}(u) \dddot{Q}^{ij}_{TT}(u)$$

where $\dot{Q}_{ij} = \mathcal{L}_t Q_{ij}$,
TT=Transverse & Traceless



First ingredient: h_{ab}

$$g_{\mu\nu}(\epsilon) = a^2(\eta)\,(\mathring{g}_{\mu\nu} + \epsilon h_{\mu\nu} + \ldots)$$



Need late time expansion instead of $1/r$ expansion!

First ingredient: h_{ab}

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + e^{2Ht}\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right) \qquad H^2 = \frac{\Lambda}{3}$$

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Gravitational field satisfies

$$\left(-\frac{\partial^2}{\partial t^2} + e^{-2Ht} D^2 - 3H\frac{\partial}{\partial t}\right)\bar{h}_{ij} = 16\pi G\, e^{-2Ht}\, T_{ij}$$

which has solutions in the late time regime [Vega, Ramirez, Sanchez; 1998]

$$\bar{h}_{ij} = \frac{4G}{r}\int \mathrm{d}^3x'\, T_{ij}(t_r, x') \underbrace{-4GH\int_{-\infty}^{t_r} \mathrm{d}t'\, e^{Ht'}\,\frac{\partial}{\partial t'}\int \mathrm{d}^3x'\, T_{ij}(t', x')}_{\text{tail term}} + \mathcal{O}\left(\frac{D_o}{\ell_\Lambda} + \frac{D_o}{d(t)}\right)$$

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Assumptions

- $D(t) \leq D_o \ll \ell_\Lambda \equiv \frac{1}{H}$
- $D_o < d(t)$ (better for late times due to expansion)
- $\mathcal{L}_T T_{ab}$ is non-zero only for a finite time-interval

Second ingredient: energy formula

Using the symplectic framework, the Hamiltonian associated with de Sitter time translation is

$$
\begin{aligned}
H_T &= -\frac{1}{2}\omega(h, \mathcal{L}_T h) \\
&= \frac{1}{16\pi G H}\int \mathrm{d}^3\mathcal{I}^+\, \mathcal{E}^{ij}\left(\mathcal{L}_T \bar{h}_{ij}^{TT} + 2H\,\bar{h}_{ij}^{TT}\right)
\end{aligned}
$$

Third ingredient: quadrupole moment I

Definition

$$Q_{ij} := \underbrace{\int \mathrm{d}^3 V}_{a^3 \mathrm{d}x\mathrm{d}y\mathrm{d}z} \; \underbrace{\rho}_{T_{\mu\nu}\partial_t^\mu \partial_t^\nu} \; \underbrace{(a\, x_i)\,(a\, x_j)}_{\text{physical distance}}$$

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Interesting feature of de Sitter:

$$\mathcal{L}_T T_{\mu\nu} = 0 \Longrightarrow \mathcal{L}_T \rho = 0$$

$$\Longrightarrow \mathcal{L}_T Q_{ij} = \cancelto{0}{\int \mathrm{d}^3 V\, \mathcal{L}_T \rho\,(a x_i)\,(a x_j)} - 2HQ_{ij} \neq 0$$

Third ingredient: quadrupole moment II

Conservation of stress-energy tensor $\bar{\nabla}^{\mu} T_{\mu\nu} = 0$ is

$$\partial_t \rho - e^{-Ht} D^i T_{0i} + 3H(\rho + P) = 0$$
$$\partial_t T_{0i} - D^j T_{ij} + 3H T_{0i} = 0$$

$$\Longrightarrow \int \mathrm{d}^3 x T_{ij} = \frac{1}{2} \int \mathrm{d}^3 V \left(\ddot{\rho} + 3He^{-Ht} \dot{\rho} + 3H^2 e^{-2Ht} \rho \right) (ax_i)(ax_j)$$
$$+ \frac{1}{2} \int \mathrm{d}^3 V \left(He^{-Ht} \dot{P} + 3H^2 e^{-2Ht} P \right) (ax_i)(ax_j)$$

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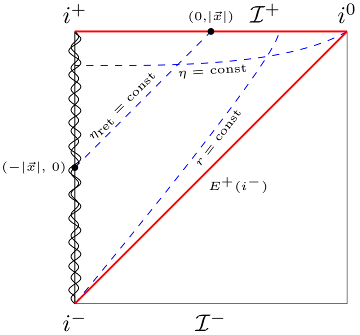
$$\boxed{\int \mathrm{d}^3 x T_{ij} = \frac{a^{-1}}{2} \left(\ddot{Q}_{ij}^{(\rho)} + 2H\dot{Q}_{ij}^{(\rho)} + H\dot{Q}_{ij}^{(P)} + 2H^2 Q_{ij}^{(P)} \right)}$$

Energy

$$H_T = \frac{G}{8\pi}\int \mathrm{d}^3\mathcal{I}^+ \left(\mathcal{R}_{ab}\mathcal{R}_{cd}^{TT}\right)_{ret} \mathring{q}^{ac}\mathring{q}^{bd}$$



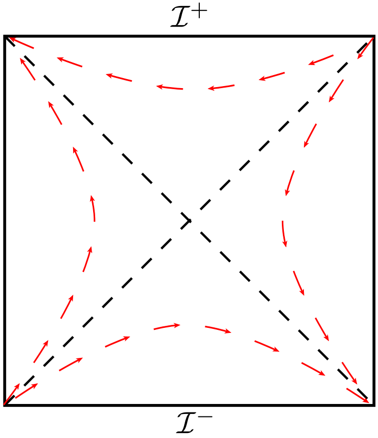
$$\mathcal{R}_{ab} = \dddot{Q}_{ab}^{(\rho)} + 3H\ddot{Q}_{ab}^{(\rho)} + 2H^2\dot{Q}_{ab}^{(\rho)} + H\ddot{Q}_{ab}^{(P)} + 3H^2\dot{Q}_{ab}^{(P)} + 2H^3 Q_{ab}^{(P)}$$

Properties energy

$$H_T = \frac{G}{8\pi} \int \mathrm{d}^3\mathcal{I}^+ \ \left(\mathcal{R}_{ab}\mathcal{R}_{cd}^{TT}\right)_{ret} \ \mathring{q}^{ac}\mathring{q}^{bd}$$

- Features information about energy density *and* pressure
- Recover result Minkowski in limit $H \to 0$
- $\mathcal{L}_T T_{ab} = 0 \Longrightarrow \dot{Q}_{ab} \neq 0$ but $H_T = 0$ (as it should!)
- Energy is positive

How does positivity arise?



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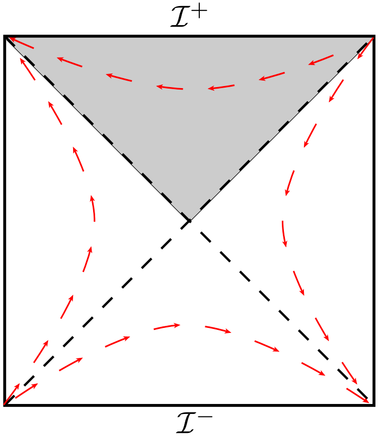

Conclusion & future work

$$P_T = \frac{G}{8\pi} \int \mathrm{d}\Sigma_2 \, \left(\mathcal{R}_{ab}\mathcal{R}^{TT}_{cd}\right)_{ret} \, \mathring{q}^{ac}\mathring{q}^{bd}$$

- Important for conceptual understanding: one step closer to a complete framework of asymptotics with $\Lambda > 0$
- Observational consequences: replace de Sitter background by a FLRW background with $\Lambda > 0$
- Any suggestions?

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THANK YOU