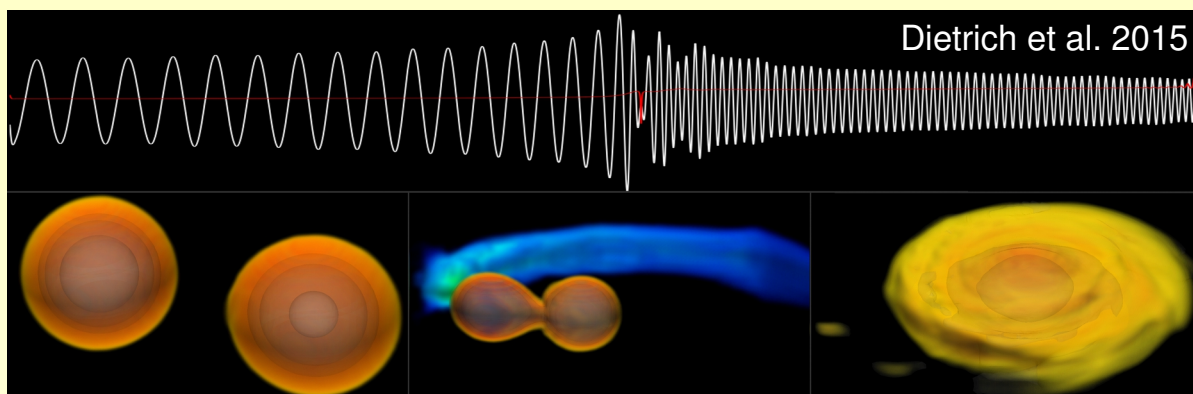Bernd Brügmann
11.7.2016
GR21, New York

Solving 3D relativistic hydrodynamical problems with WENO discontinuous Galerkin methods

Dietrich et al. 2015

Jena/BAM Collaboration

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Bernd Brügmann, Jena
…

Outline:
   1. Numerical methods for GR and GR-hydro
   2. Code project BAMPS
   3. WENO DG method for a TOV star

Wanted:
General purpose numerical method for general relativity and relativistic astrophysics

Multi-physics
 black holes, neutron stars,
   gravitational waves, EM counterparts, ...
 Einstein equations, GR-hydro, …
 smooth solutions, shocks

Formulation
 stability, constraints, gauge, initial data
 non-conservative, conservative

**> 30 years of numerical relativity
 and computational astrophysics
 … ongoing!**

Efficiency
 efficiency = physics / cost
 physics = features, accuracy
 cost = code-time, run-time,
              memory, dollars

High-order convergence
 6th-10th order FD
 5th high-res. shock capturing
 spectral methods

Adaptive mesh refinement
 in space and time

Parallelization
 $10^3 \rightarrow 10^6$ cores

Discretization of PDEs

Example: 1d non-linear conservation law

$$u(t, x),\ x \in [-1, 1]: \qquad \boxed{\partial_t u + \partial_x f(u) = 0}$$

Nodal collocation in space (as opposed to modal)

$$x \rightarrow x_i, \quad u \rightarrow u_i = u(t, x_i), \quad f(u) \rightarrow f_i = f(u_i), \quad i = 0, \ldots, N$$

Finite difference, pseudo spectral, discontinuous Galerkin

$$
\begin{aligned}
FD: & \quad \partial_t u_i + D_{ij}^{\mathsf{FD}} f_j &=& \quad 0 \\
PS: & \quad \partial_t u_i + D_{ij}^{\mathsf{PS}} f_j &=& \quad 0 \\
DG: & \quad \partial_t u_i + D_{ij}^{\mathsf{DG}} f_j &=& \quad [g_i^*]_{-1}^{+1}
\end{aligned}
$$

- differentiation matrices are closely related, e.g.
  $$D^{\mathsf{DG}} = M^{-1} D^{\mathsf{PS}} \approx D^{\mathsf{PS}}$$

- cell boundaries, modulo details:

  $$\boxed{\text{DG with numerical flux } g^* \Longleftrightarrow \text{PS with specific penalty method}}$$

Spectral element methods:
pseudospectral penalty and discontinuous Galerkin methods

Features:
- exponential convergence (and $O(h^{k+2})$ for C^k)
  - $\rightarrow$ often optimal in accuracy/memory
- finite-size multipatch or h-size elements/cells
  - $\rightarrow$ hp-refinement, t-refinement; in principle more efficient than multipatch
- (!) discontinuous cell coupling
  - $\rightarrow$ derivatives are cell-local (cmp. FD)
  - $\rightarrow$ efficient parallelization, essential for trend to many-core systems

Open or under development:
- DG for GR? DG for GR-hydro?
  - $\rightarrow$ (!) PS for GR and DG for hydro with same grid and operators
  - $\rightarrow$ (!) evaluate WENO DG etc. for shocks (little known so far)

Discontinuous Galerkin for General Relativity

Zumbusch (2009):
first/only example for full 3D Eeq., space-time DG

Field, Hesthaven, Lau, Mroue (2010); Brown, Diener, Field, Hesthaven, Hermann, Mroue, Sarbach, Schnetter, Tiglio, Wagman (2012):
second and first order BSSN, mostly 1D

Radice, Rezzolla (2011):
general discussion of DG for matter, 1D example

Teukolsky (2016):
formulation of discontinuous Galerkin for relativistic astrophysics
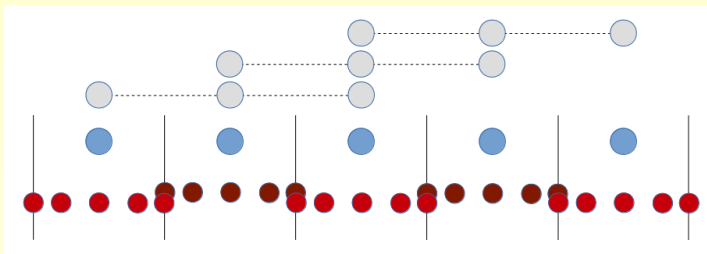
Miller, Schnetter (2016):
second order BSSN in vaccum, 3D

Group efforts 2016:
Field/SXS et al, Rezzolla et al, Brügmann et al

Discontinuous Galerkin with shock capturing

Qiu, Shu (2005), (2005):   WENO DG method, troubled cell indicators
Zhao, Tang (2013):   first application of WENO DG to special relativistic hydrodynamics
Zhong, Shu (2013):   simple WENO (reduced stencil size)
Dumbser, Zanotti, et al (…,2014,…); Sonntag, Munz (2014); Huerta, Casoni, Peraire (2012):
                          hybrid DG/FV subcell method
                          replaces troubled cells with equidistant FV method
                          in part using space+time Galerkin

Catch: Subcell resolution?!? Cell-locality?

WENO5 = Sum 3 x 3-point-stencils
cell averages
DG grid, 5 points per cell

standard WENO-DG uses cell averages for WENO scheme
e.g. simple WENO5 needs 27, WENO5 needs 125 times more resources
    competitive when used only for troubled cells and with AMR?

BAMPS code, based on BAM (since 1997)

Pseudospectral method for a
single black hole (on a GPU)
BB, J. Comp. Phys. 235 (2013)

Pseudospectral method for
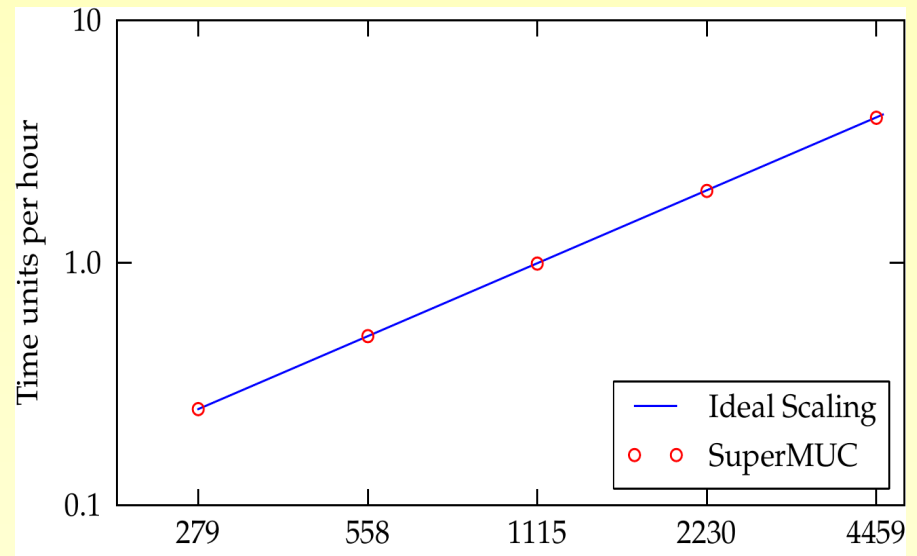gravitational wave collapse
Hilditch, Weyhausen, BB, PRD (2016)

Solving 3D relativistic hydro
problems with WENO discontinuous
Galerkin methods (e.g. TOV star)
Bugner, Dietrich, Bernuzzi, Weyhausen,
BB (2015) arXiv:1508.07147

Key features:
  high-order convergence
  efficient parallelization

Hyperboloidal slicing for GR:
mathematical analysis and
wave equation tests
Hilditch, Harms, Bugner, Rüter, BB,
in preparation (2016)

Work in progress:
  full GR and GR-hydro
  hyperboloids for GW at Scri
  AMR in space and time
  AMR parallelization

Discontinuous Galerkin for GR-hydrodynamics

Bugner, Dietrich, Bernuzzi, Weyhausen, BB (2015) arXiv:1508.07147

- Compare
  standard WENO FV/FD (BAM)
  WENO3, WENO5, WENO5Z for DG
  simple WENO5 for DG
  WENO DG/FV subcell

- Consider
  standard 1d tests
  SRHD in 1d and 2d (cmp. Zhong, Shu 2013, recently Field et al 2016)
  GR-hydro + GR (Cowling approximation) for TOV star in 1d, 2d, 3d
  first application of DG to GR and GR-hydro

SRHD in 1d: shock tube

comparisons with
S. Field et al.

Special relativistic shock tube in 1D
density (red), velocity (blue), pressure (green)
WENO3, WENO5, WENO5Z, simple WENO5, WENO subcell
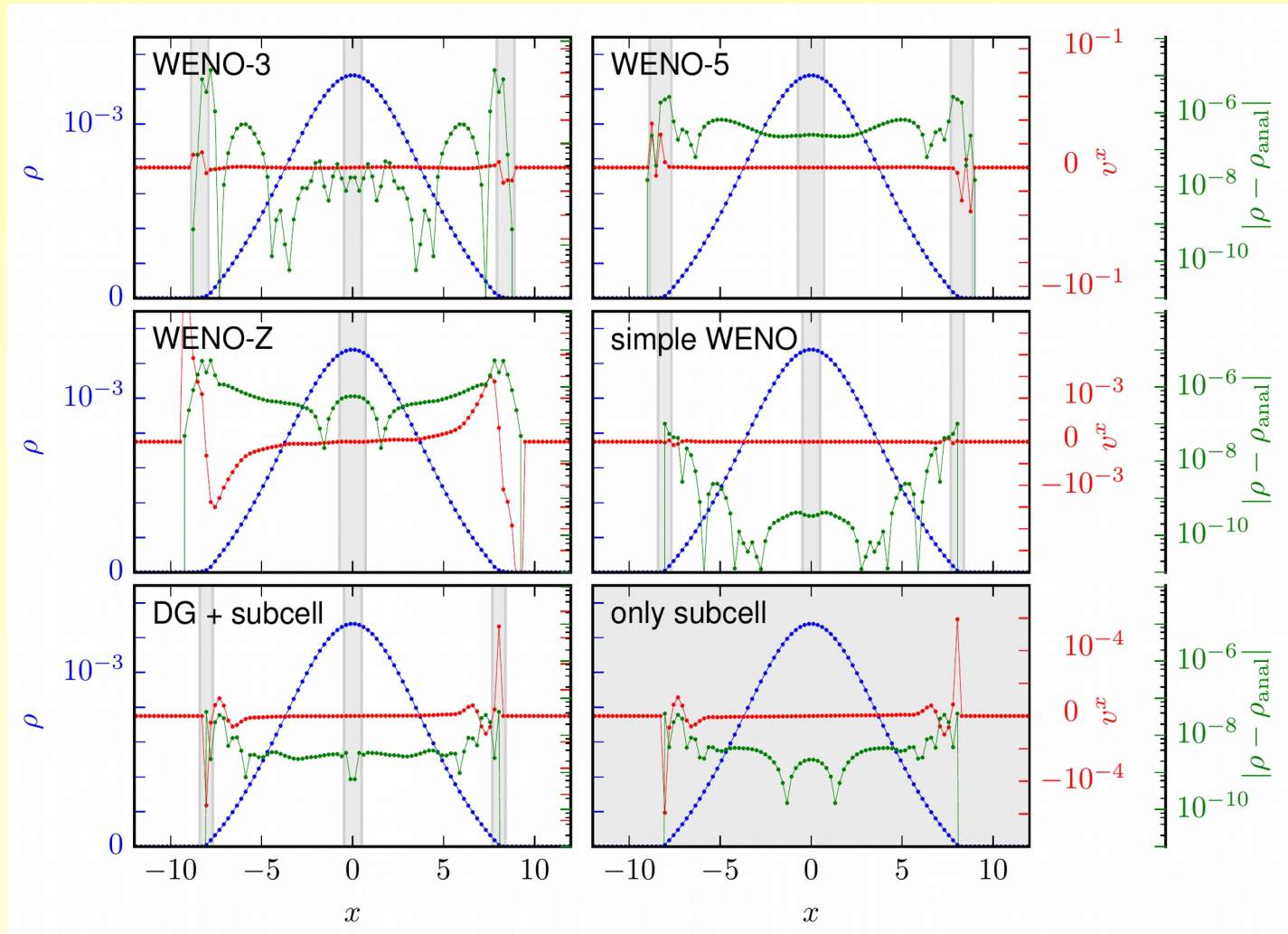
SRHD in 2D

Shock tube and vortex
WENO5 vs. subcell

density, velocity
100x100 elements
troubled cells shaded

both methods stable
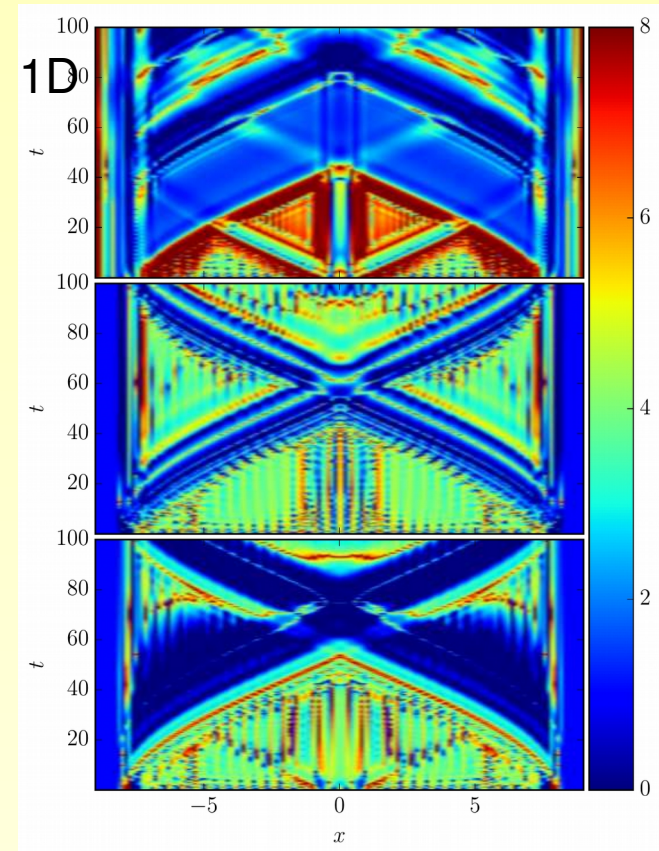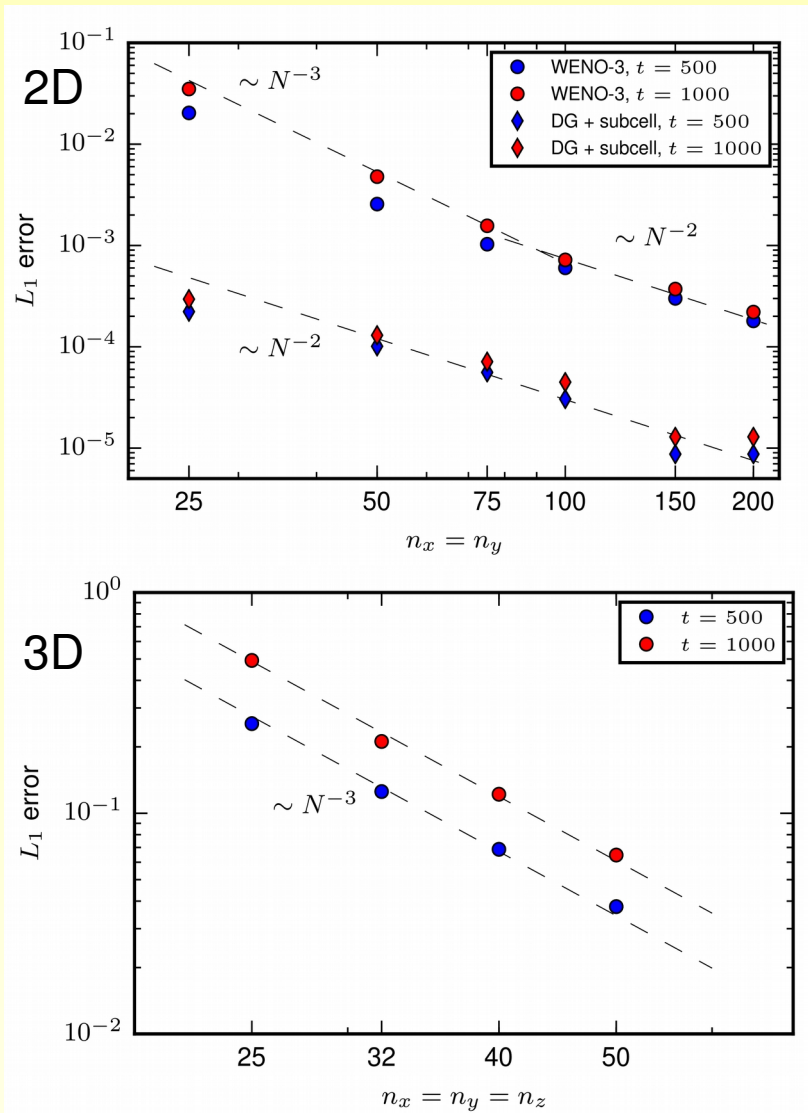subcell more accurate
at higher costs

TOV star in 1D

t = 1000, 100 cells, troubled shaded, Γ=2 polytrope, atmosphere 1e-8

B. Brügmann, Theoretical Physics Institute, University of Jena   http://www.tpi.uni-jena.de
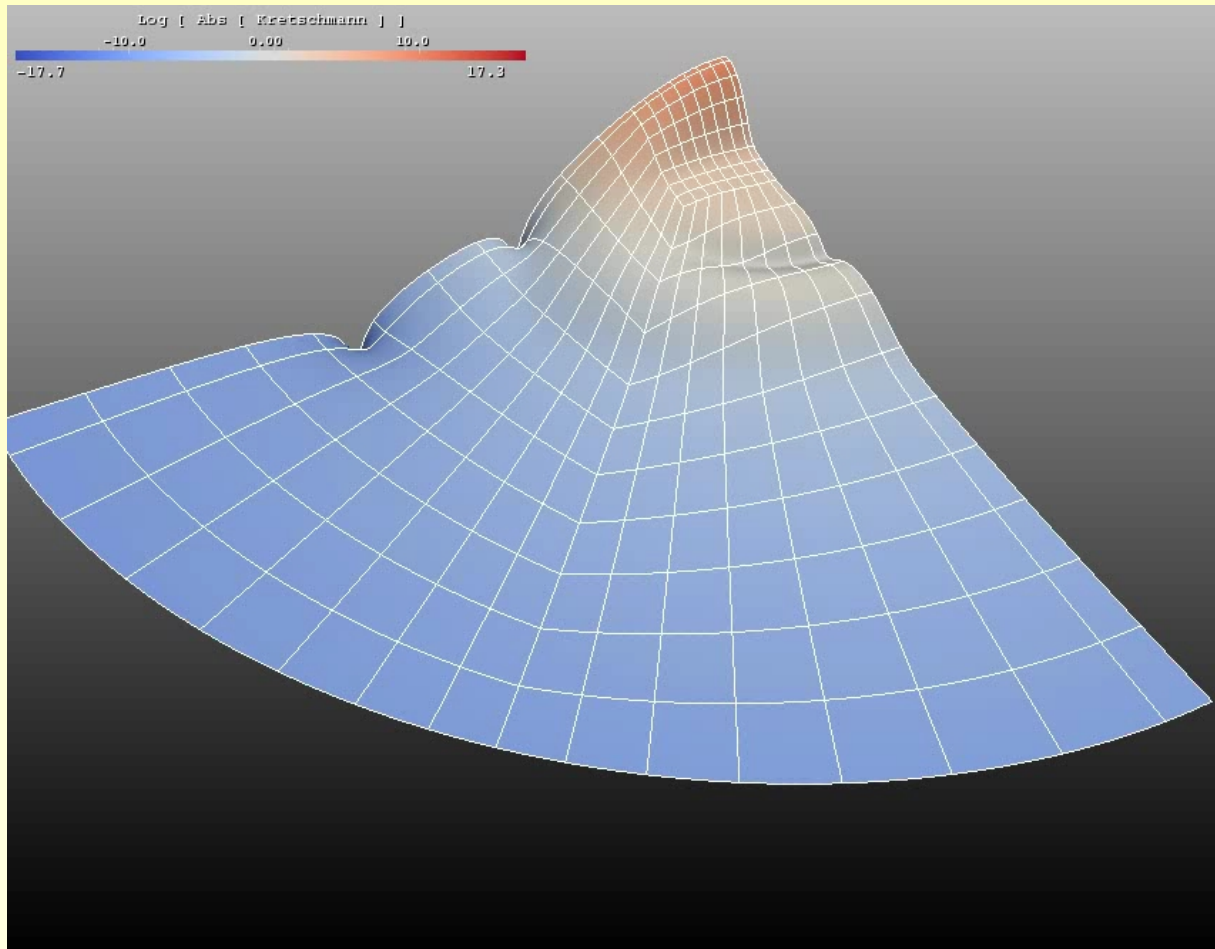
TOV star: convergence

WENO3, simple WENO5, subcell
Does non-convergence spread?
~ sqrt(t) in Cockburn, Guzman 2008
worse for artificial viscosity in
Klöckner, Warburton, Hesthaven 2011

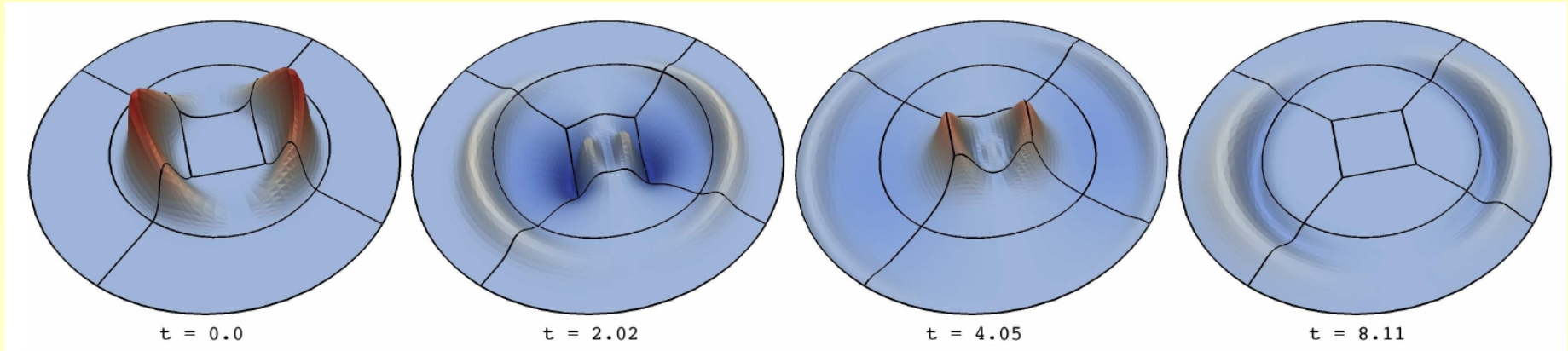# Collapse of Gravitational Waves in 2d and 3d

BAMPS: Hilditch, Weyhausen, BB, PRD (2016)
BAM: Hilditch, Baumgarte, Weyhausen, Dietrich, BB, Montero, Müller, PRD (2013)

Hyperboloidal slices extending to null infinity

Hilditch, Harms, Bugner, Rüter, BB, in preparation (2016)

Numerical example: scalar wave with quadrupolar initial data
New formulation for full 3D (vacuum) general relativity
Uses Hilditch (2015) dual foliation framework
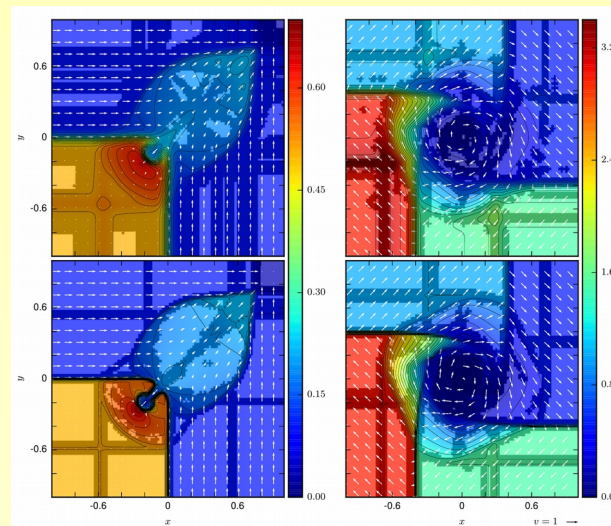PS method implemented in BAMPS

Summary

Numerical general relativity and
relativistic astrophysics continue to
be computationally highly challenging.

The discontinuous Galerkin method
is a promising new paradigm for PDEs:
high order, hp-refinement, efficient
parallelization. Key question: competitive
for non-smooth solutions?

First DG simulations of a 3D TOV star
in GR-hydro (Cowling).

BAMPS infrastructure, work in progress:
  full GR and GR-hydro
  hyperboloids for GW at Scri
  AMR in space and time
  AMR parallelization

Case for spectral elements:

• If smooth, use DG/PS:

    GR ✓   GRhydro ✓

• If not smooth:

    use DG/FV locally

    use DG/PS where smooth

    overall efficiency ?