New charges for BMS symmetries

[in Dissertation]

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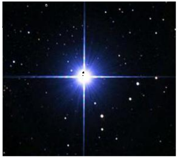
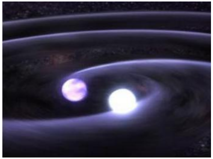
With Abhay Ashtekar

Motivation

How can General Relativity be used to study systems in the universe?



SINGLE STAR



BINARY NEUTRON STAR

- Astrophysics: Properties, gravitational waves
- Mathematical Physics: Positivity of energy
- Quantum theory: Evaporation of black holes

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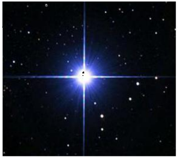

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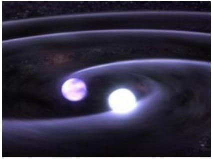

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- Einstein's equations are **second order and non-linear.** Exact solutions are rare, highly idealized.

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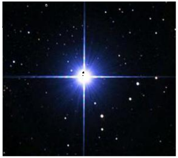

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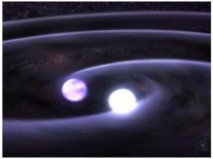

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Can we recover some structure for isolated systems?

Asymptotics 101

Yes!

Key insight: [1] Far away from a gravitating source, *space-time becomes flat.*

[1] Arnowitt, Deser, Misner, Bondi, van der Burgh, Metzner, Sachs, Penrose, Geroch and others

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Asymptotically flat space-times [1]

1. Define *isolated system*. Supply boundary conditions.
2. Obtain *asymptotic symmetries* as diffeomorphisms which preserve boundary conditions.
3. Find *conserved quantities* and *conservation laws*.
   3.1. Characterization of gravitational radiation, and,
   3.2. Well-defined notions of physical observables.

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Missing: Tensorial expression for conserved quantities. We provide it.

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1. Asymptotically flat spacetime ($\Lambda = 0$)

A spacetime $(\hat{M}, \hat{g}_{ab})$ is **asymptotically flat at null infinity** [2] if $\exists$ a manifold (M, g_{ab}) such that

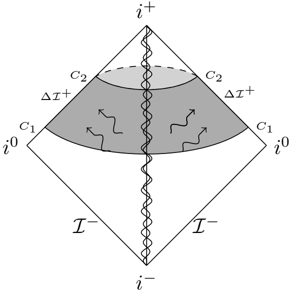
1. **Conformal completion:** *one can attach a boundary 'at infinity'*, $\hat{M}$ is an open submanifold of M with smooth boundary $\partial M = \mathcal{I}$, $\exists$ a smooth scalar field Ω on M such that $g_{ab} = \Omega^2 \hat{g}_{ab}$ on $\hat{M}$, and $\Omega = 0$ & $\mathrm{d}\Omega \neq 0$ on $\mathcal{I}$,
2. **Boundary conditions**: *at this boundary matter fields 'fall-off sufficiently fast'*,
   $\hat{G}_{ab} = 8\pi G\, \hat{T}_{ab}$ where $\Omega^{-2}\, \hat{T}_{ab}$ has a smooth limit to $\mathcal{I}$. $\Omega^{-2} \sim r^{-2}$
3. Topology of $\mathcal{I}$ is $\mathbb{S}^2 \times \mathbb{R}$. $\mathcal{I}$ is complete.
   Examples: Minkowski, Kerr etc.

[2] Penrose, Geroch, ...

2. Consequences

(i) $\mathcal{I}$ is **Null.**

- $n^b := g^{ab}\nabla_b\Omega$: normal to $\mathcal{I}$
- EE $\implies$ $n.n \hat{=} 0$, $q_{ab}n^b \hat{=} 0$.



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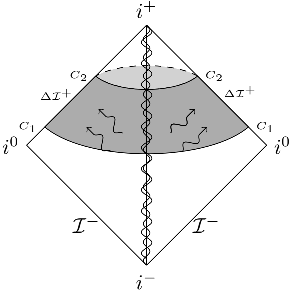


(ii) ASG is the **BMS group**.

- On $\mathcal{I}$, conformal class (q_{ab}, n^a)
  q_{ab}: intrinsic metric on $\mathcal{I}$,
- *Asymptotic symmetries*, ξ^a,
  $\mathcal{L}_\xi q_{ab} = 2\,k\,q_{ab}; \quad \mathcal{L}_\xi n \hat{=} -k\,n^a$
- *Bondi-Metzner-Sachs (BMS) group* $\simeq$ *ST* $\ltimes$ *Lorentz*
  ST: Supertranslations
  $\xi^a = \alpha n^a$, $\mathcal{L}_n\alpha = 0$

  General BMS field
  $\xi^a = \alpha(\theta, \phi)\, n^a + v^a$

3.1. Gravitational radiation at $\mathcal{I}$

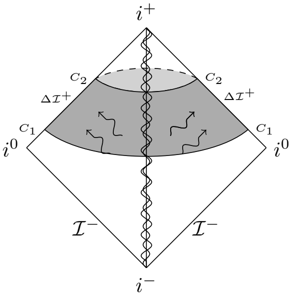
Conformally-invariant part of curvature: $N_{ab} = \mathcal{S}_{ab} - \rho_{ab}$
Gravitational 'News' tensor contains radiative information.

- S_{ab} is the Schouten tensor of the derivative operator D on $\mathcal{I}$
  $\mathcal{R}_{abc}{}^{d} = \mathcal{S}_{c[a}\delta_{b]}{}^{d} + q_{c[a}\mathcal{S}_{b]}{}^{d}$
  $\mathcal{S}_{ab}$ has complicated conformal behaviour.
- Unique symmetric tensor, ρ_{ab}, s.t. **[Geroch]**
  $\rho_{ab}n^{b} = 0, \quad \rho_{ab}q^{ab} = \mathcal{R}, \quad D_{[a}\rho_{b]c} = 0$
  Same conformal transormation as $\mathcal{S}_{ab}$.
- $N_{ab} = 0$ reduces BMS to Poincaré group.

3.2. Conservation Laws: Asymptotic Fluxes

Symmetries $\rightarrow$ Conservation Laws

- Every symmetry ξ is associated with a **Flux** $F_\xi[\Delta\mathcal{I}]$
- Time translation - energy, rotation - angular momentum.
- **Asymptotic flux** **[Ashtekar and Streubel]**



$$F_\xi[\Delta\mathcal{I}] = \frac{1}{2\kappa}\int_{\Delta\mathcal{I}} dV\, N_{cd}\,[(\mathcal{L}_\xi D_a - D_a\mathcal{L}_\xi)\ell_b + 2\,\ell_{(a}D_{b)}k]\,q^{ac}\,q^{bd}$$

where $\kappa = 8\pi G$, ℓ_a is the null covector orth to n^a, $\ell.n = -1$, $k = D_a\xi^b$.

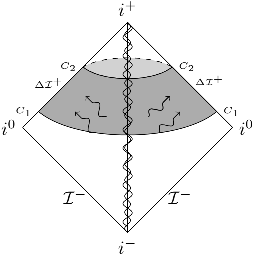
3.2. Conservation Laws: Asymptotic Fluxes (contd.)

Can we rewrite the flux using Stoke's theorem as an integral of 'time' derivative of a charge?

$$F_\xi[\Delta\mathcal{I}] = \int_{\Delta\mathcal{I}} \mathrm{d}\mathcal{Q}_\xi = Q[C_1] - Q[C_2]$$

where $Q[C_1] = \oint_{C_1} \mathcal{Q}_\xi$ is the instantaneous charge.

Yes! **[Wald and Zoupas]**



Done for supertranslations **[Geroch, Ashtekar and Streubel]**, extended to Lorentz transformations **[Dray and Streubel]** but NP notation obscures tensorial structure.

We propose a new charge expression.

Charges of BMS symmetries

$$Q_\xi[C] = -\frac{1}{\kappa}\oint_C \mathrm{d}S\Big[\{K_{abc}{}^d\,\ell^a\,n^c + (\mathcal{R}_{abc}{}^d - \mathring{R}_{abc}{}^d)\bar{q}^{am}\,\bar{q}^{cn}D_m\ell_n\}\,\xi^b\ell_d$$
$$-\frac{1}{6}(D_p\xi^p)\,(D_a\ell_b)\,(D_c\ell_d)\,\bar{q}^{bd}\,\bar{q}^{cm}\Big]$$

where $K_{abc}{}^d = \Omega^{-1}C_{abc}{}^d$, $\mathcal{R}_{abc}{}^d = \mathcal{S}_{c[a}\delta_{b]}{}^d + q_{c[a}\mathcal{S}_{b]}{}^d$,
$\mathring{R}_{abc}{}^d = \rho_{c[a}\delta_{b]}{}^d + +q_{c[a}\rho_{b]}{}^d$

Salient features

- Linear in BMS field.
- Makes structure clear, conformally invariant.
- Yields the Ashtekar-Streubel asymptotic flux.
- Not tied to null tetrad defined in space-time. Agrees with Dray and Streubel's charge when one is chosen.

Charge to flux I: Supertranslations

Supertranslations: $\xi^a = \alpha n^a$; $\mathcal{L}_n \alpha = 0$

Main ingredients

- **Simplify Ashtekar-Streubel Flux**
  $(\mathcal{L}_\xi D_m - D_m \mathcal{L}_\xi)\ell_n = \xi^c (q_{n[c}\mathcal{S}_{m]}{}^d + \mathcal{S}_{n[c}\delta_{m]}{}^d)\ell_d - \ell_d D_m D_n \xi^d$

$$F_\xi[\Delta\mathcal{I}] \;\; = \;\; \frac{1}{\kappa}\int_{\Delta\mathcal{I}} \mathrm{d}^3V \Big[\frac{1}{4}\alpha\, N_{ac}\,\bar{q}^{am}\,\bar{q}^{cn}\,\mathcal{S}_{mn} + \frac{1}{2}\,N_{ac}\,\bar{q}^{am}\,\bar{q}^{cn}\,D_m D_n \alpha\Big]$$

Goal: $F_\xi \equiv \int$ 'Time derivative' of new charge

$$\int_{\Delta\mathcal{I}} \mathcal{L}_n Q = \frac{1}{\kappa}\int_{\Delta\mathcal{I}} \mathrm{d}^3V\,\mathcal{L}_n[\alpha\, K_{abcd}\ell^a n^b \ell^c n^d - \frac{1}{2}\alpha\, N_{ac}\,(D_m\ell_n)\,\bar{q}^{am}\,\bar{q}^{cn}]$$

Charge to flux I: Supertranslations (contd.)

$$F_\xi[\Delta\mathcal{I}] \;=\; \frac{1}{\kappa}\int_{\Delta\mathcal{I}} \mathrm{d}^3V\Big[\frac{1}{4}\alpha\, N_{ac}\,\bar{q}^{am}\,\bar{q}^{cn}\,S_{mn} + \frac{1}{2}\,N_{ac}\,\bar{q}^{am}\,\bar{q}^{cn}\,D_mD_n\alpha\Big]$$

$$\int_{\Delta\mathcal{I}}\mathcal{L}_nQ = \frac{1}{\kappa}\int_{\Delta\mathcal{I}}\mathrm{d}^3V\,\mathcal{L}_n[\alpha\,K_{abcd}\ell^a n^b\ell^c n^d - \frac{1}{2}\alpha\,N_{ac}\,(D_m\ell_n)\,\bar{q}^{am}\,\bar{q}^{cn}]$$

- **Bianchi identities**

$$\begin{aligned}
\mathcal{L}_n(K_{abcd}\ell^a n^b\ell^c n^d) &\;\hat{=}\; \bar{q}^{fd}\nabla_f K_{abcd}n^a\ell^b n^c\\
\mathcal{L}_nD_m\ell_n &\;\hat{=}\; -\frac{1}{2}[q_{mn}\mathcal{S}_c{}^d n^c\ell_d + (n.\ell)\mathcal{S}_{mn}]\\
\mathcal{L}_nN_{ac} &\;\hat{=}\; -2K_{racs}n^r n^s
\end{aligned}$$

Using Bianchi identitities and Integration By Parts, the two match!

$$F_\xi[\Delta\mathcal{I}] = Q_\xi[C_1] - Q_\xi[C_2]$$

Charge to flux II: Lorentz transformations

Lorentz transformations: $\xi^a = v^a$

Main ingredients:

- **Simplify Ashtekar-Streubel Flux**
- **Bianchi identitites**, Integration By Parts

$$F_\xi[\Delta\mathcal{I}] - \int_{\Delta\mathcal{I}} \mathcal{L}_n Q_\xi$$
$$= \int_{\Delta\mathcal{I}} \mathrm{d}V \left[(D_a D_b k + \frac{1}{2}\mathcal{L}_h \rho_{ab}) D_c \ell_d \, \bar{q}^{ac} \, \bar{q}^{bd} \right]$$

where $\mathcal{L}_\xi q_{ab} = 2\, k\, q_{ab}$.

- Trace-free symmetric part of $D_a D_b k + \frac{1}{2}\mathcal{L}_v \rho_{ab}$ is zero for a BMS field.

$$F_\xi[\Delta\mathcal{I}] = Q_\xi[C_1] - Q_\xi[C_2]$$

Summary

- New tensorial expression for general BMS field on arbitrary cross-section of $\mathcal{I}$.
- Linear in BMS field.
- Yields the Ashtekar-Streubel Hamiltonian flux, making role of Bianchi identities more transparent. Two cross-sections must be related by supertranslations.
- Not tied to null tetrad defined in spacetime. Agrees with Dray and Streubel's charge when null tetrad is chosen. Reduces to Geroch's supermomentum for supertranslations.
- Explore connections with charges of extended BMS

  [Flanagan and Nichols, Strominger et. al., Campiglia and Laddha]

THANK YOU!