On Stability of the Charged Brans-Dicke Space-times

Presentation by A Lun in GR21

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This talk consists of two parts:

Part 1:
Static Spherically Symmetric Charged Brans-Dicke (CBD) Space-times

Part 2:
Time-independent perturbation of the Maxwell Equations in CBD background Space-times: Exact solution of a toy model.

Part 1: Static Spherically Symmetric Electrov Brans-Dicke Space-times

- We extended the four branches of the Brans-Dicke solutions (Brans 1961) to include a static isolated electric charge as source.
- Recall the four branches of the Brans solutions (Brans 1962) are:

$ds^2 = -c^2 e^{2\alpha(r)} dt^2 + e^{2\beta(r)} \left[dr^2 + r^2 \left(d2\right) + \sin^2\theta d\varphi^2\right]$				
Brans parameter C and $\lambda^2 := \frac{\omega+2}{2} C^2 + C + 1$				
	$\frac{e^{2\alpha(r)}}{e^{2\alpha_0}}$	$\frac{e^{2\beta(r)}}{e^{2\beta_0}}$	$\frac{\phi(r)}{\phi_0}$	λ^2
I	$\left\lvert\frac{1-\frac{B}{r}}{1+\frac{B}{r}}\right\rvert^{\frac{2}{\lambda}}$	$\left(1-\frac{B^2}{r^2}\right)^2 \left\lvert\frac{1-\frac{B}{r}}{1+\frac{B}{r}}\right\rvert^{-\frac{2(C+1)}{\lambda}}$	$\left\lvert\frac{1-\frac{B}{r}}{1+\frac{B}{r}}\right\rvert^{\frac{C}{\lambda}}$	> 0
II	$e^{\frac{4}{\sqrt{-\lambda^2}} \tan^{-1}\frac{B}{r}}$	$\left(1-\frac{B^2}{r^2}\right)^2 e^{-\frac{4(C+1)}{\sqrt{-\lambda^2}} \tan^{-1}\frac{B}{r}}$	$e^{\frac{2C}{\sqrt{-\lambda^2}} \tan^{-1}\frac{B}{r}}$	< 0
III	$e^{-\frac{2r}{B}}$	$\frac{B^4 e^{\frac{2(C+1)r}{B}}}{r^4}$	$e^{-\frac{Cr}{B}}$	$= 0$
IV	$e^{-\frac{2}{Br}}$	$e^{\frac{2(C+1)}{Br}}$	$e^{-\frac{C}{Br}}$	$= 0$

CBD solutions (continue ...1)

- The different branches of the Brans solutions arise from the representations of a complex function $\ln z = \ln |z| + i \arg z$, where $|z| = 1$.
- Specifically, they arise from a real function of a complex variable (see also Bhadra and Sarkar 2005)

$$\exp\left(\frac{\phi_1}{2\sqrt{ab}} \ln\left(\frac{\sqrt{\frac{a}{b}}r - 1}{\sqrt{\frac{a}{b}}r + 1}\right)\right),$$

  where the real constants a, b and ϕ_1 are given in terms of the Brans constants B, C and λ such that $\frac{C^2}{\lambda^2} := \frac{(\phi_1)^2}{4ab}$ and $B^2 = \frac{b}{a}$.
- Written in familiar form: $B^2 = M^2 := \frac{b}{a} > 0$, in terms of the mass parameter M.
- In the CBD solutions, $B^2 \to M^2 - Q^2$, which is allowed to become negative when charge is present.

CBD solutions (continue ...2)

- In the CBD space-times, the Brans parameter $C = 2\left(-1+2\sqrt{\kappa}\right)^{-1}$ (same as BD) is allowed to be complex due to the presence of an isolated electric charge. (We thank Carl Brans for his thesis.)
- The CBD spacetimes are given formally

$ds^2 = -c^2 e^{2\alpha(r)} dt^2 + e^{2\beta(r)}\left[dr^2 + r^2\left(d2\right) + \sin^2\theta d\varphi^2\right]$	
CBD parameter $\kappa = \frac{4ab}{(\phi_1)^2} - \frac{2\omega+3}{4} = \left(\frac{1}{C}+\frac{1}{2}\right)^2$	
$\frac{e^{2\alpha(r)}}{e^{2\alpha_0}}$	$\exp\left(-\frac{\phi_1}{2\sqrt{ab}}\ln\left(\frac{\sqrt{\frac{a}{b}}r-1}{\sqrt{\frac{a}{b}}r+1}\right)\right)\frac{1}{N^2(r)}$
$\frac{e^{2\beta(r)}}{e^{2\beta_0}}$	$\frac{1}{\phi_0}\left(1-\frac{b}{ar^2}\right)^2\exp\left(-\frac{\phi_1}{2\sqrt{ab}}\ln\left(\frac{\sqrt{\frac{a}{b}}r-1}{\sqrt{\frac{a}{b}}r+1}\right)\right)N^2(r)$
$\frac{\phi(r)}{\phi_0}$	$\exp\left(\frac{\phi_1}{2\sqrt{ab}}\ln\left(\frac{\sqrt{\frac{a}{b}}r-1}{\sqrt{\frac{a}{b}}r+1}\right)\right)$
$V'(r)$	$\frac{c^2 Q\phi_1}{\sqrt{4\pi}}\frac{e^{2\alpha_0}\phi_0{}^{\text{`}}}{ar^2\left(1-\frac{b}{ar^2}\right)N^2(r)}$
$N(r)$	$p_+^2 e^{-\frac{\phi_1\sqrt{\kappa}}{2\sqrt{ab}}\ln\left(\frac{\sqrt{\frac{a}{b}}r-1}{\sqrt{\frac{a}{b}}r+1}\right)} - p_-^2 e^{\frac{\phi_1\sqrt{\kappa}}{2\sqrt{ab}}\ln\left(\frac{\sqrt{\frac{a}{b}}r-1}{\sqrt{\frac{a}{b}}r+1}\right)}$

CBD solutions (continue ...3)

- The constants

$$p_{\pm}^2 = \frac{1}{4}\left(\sqrt{1+\frac{e^{2\alpha_0}\phi_0 Q^2}{\kappa}} \pm 1\right)$$

- The formal CBD solution has 12 branches: each of the four main branches corresponding to those in the BD solutions (where $\frac{b}{a} > 0, < 0, = \infty, = 0$) can be further split into 3 subclasses (corresponding to $\kappa := \frac{4ab}{(\phi_1)^2} - \frac{2\omega+3}{4} > 0, < 0, = 0$).
- Details of these 12 solutions are given in the PhD thesis by Maya Watanabe (2016)

Part 2: Time-independent perturbation of the Maxwell Equations in CBD background Space-times: exact solution of a toy model.

- Consider the electrostatic field of an electric point charge placed at a point $(r, \theta, \phi) = (b_0, \theta_0, 0)$ in a CBD background space-time.
- **Caveat**: This is a toy model because in a CBD background such perturbations couple back to the background geometry. Although this is perfectly fine for BD backgound perturbations. (We thank Bob Wald pointing this out.)
- The linearly perturbed electrostatic $V_B\left(r, \theta\right)$ potential satisfies the Laplace-type equation

$$\left(\triangle + \left[-\alpha'\left(r\right) + \beta'\left(r\right)\right] \frac{\partial}{\partial r}\right) V_B\left(r, \theta\right) = 0,$$

$$\triangle : \quad = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2}{\partial \varphi^2}.$$

Exact solution of a toy model (continue ...1)

- Whittaker (1927), Cohen and Wald (1971), Hanni and Ruffini (1973) used the method of separation of variables and independently found identical convergent series solution to the electrostatic perturbation equation in a Schwarzschild background space-time expressed in the usual Schwarzschild coordinates.
- They used a "*no pertubed charge inside the event horizon*" as boundary condition for their infinite series.
- They demonstrated the stability of the Schwarzschild solution under such linear perturbations.

Exact solution of a toy model (continue ...2)

- Copson (1928), using Hadamand's theory of elementary solutions (Hadamand 1923), constructed an exact solution to the electrostatic perturbation in a Schwarzschild background space-time in isotropic coordinates by applying a "*symmetric boundary condition.*"
- The event horizon of the Schwarzschild background solution in isotropic coordinates is a surface of inversion, so a "symmetric boundary condition" is easy to implement.
- He found that when tranforming his elegent exact solution expressed in isotropic coordinates, it becomes "rather complicated" in the ususal Schwarzschild coordinates.
- He found the infinite series of his transformed exact solution differs from the Whittaker's 1927 series solution by a zeroth-order term.

- Linet (1976) was able to express Copson's exact solution neatly in the usual Schwarzschild coordinates and concluded that Copson's solution was not for one but two charges, the second residing within the horizon.
- Watanabe and Lun (2013) examined the perturbation equation in a Brans-Dicke Reissner-Nordström (BDRN) background using isotropic coordinates.
- We extend the 2013 paper to the electrostatic perturbation of the CBD space-times by working with the formal solutions (see above):
    1. We calculated the Hadamand series by direct substituting $V_B\left(r,\theta\right) = \sum_{n=0}^{\infty} \frac{U_n(r)\Gamma^n(r,\theta)}{\Gamma^{\frac{1}{2}}}$ into the perturbation equation; $\Gamma\left(r,\theta\right) = r^2 + b_0^2 - 2b_0 r\cos\theta$ is the square distance between the perturbation charge and a field point.

Exact solution of a toy model (continue ...4)

2. We find the general $U_n(r)$ term:

$$U_n(r) = \prod_{m=1}^{m=n} \frac{\left(\frac{3}{2^2} - m(m-1)\right)\varepsilon + \frac{b_0^2\phi_1^2(2\omega+3)}{2^2a^2}}{m\left(m-\frac{1}{2}\right)\left(ab_0^2 - b\right)^n} \frac{b_0^{2n}U_0(r)}{\left(ar^2-b\right)^n},$$

$$U_0(r) = \frac{N(b_0)\left(ab_0^2-b\right)^{\frac{1}{2}}}{b_0} \frac{r}{N(r)\left(ar^2-b\right)^{\frac{1}{2}}},$$

and prove of the convergence of the Hadmamad series.

3. We applied Copson's method to write the perturbation potential as:

$$V_B(r,\theta) = \frac{e^{\alpha_0}\phi_0^{\frac{1}{2}} r}{N(r)\left(ar^2-b\right)} F(\gamma),$$

$$\gamma(r) = \frac{ab}{\left(ab_0^2-b\right)} \frac{\Gamma(r,\theta)}{\left(ar^2-b\right)}.$$

4. The function $F(\gamma)$ satisfies an ODE

$$\gamma(\gamma+1)F''(\gamma)+\frac{3}{2}(2\gamma+1)F'(\gamma)+\left(1+\frac{\phi_1^2\kappa}{4ab}\right)F(\gamma)=0,$$

   which has an exact solution

$$F(\gamma)=\frac{\phi_1\sqrt{\kappa}}{2\sqrt{ab}\sqrt{\gamma}\sqrt{\gamma+1}}\left[\begin{array}{c} W_1\left(\sqrt{\gamma+1}+\sqrt{\gamma}\right)^{\frac{\phi_1\sqrt{\kappa}}{\sqrt{ab}}} \\ -W_2\left(\sqrt{\gamma+1}-\sqrt{\gamma}\right)^{\frac{\phi_1\sqrt{\kappa}}{\sqrt{ab}}}\end{array}\right].$$

5. Using Gauss divergent theorem we impose the single perturbative charge condition and show the constants are proportional to

$$\frac{W_1}{W_2}=-\frac{p_+^2}{p_-^2}.$$

References mentioned in the slides, for detail references see Watanabe (2016):

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