Extremal black hole initial data deformations

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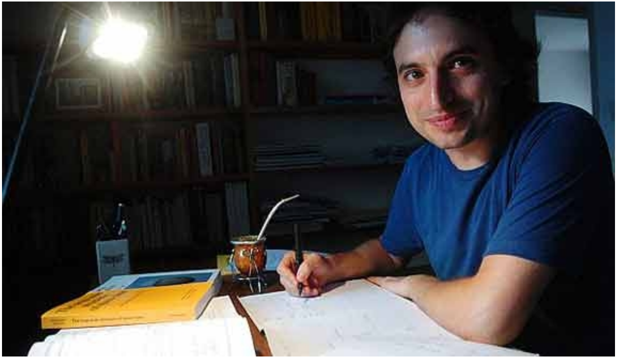
In memoriam



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Outline

- Introduction
- Field equations
- Main result
- Steps of the proof
- Conclusions

Introduction

- Big interest: understand initial data with cylindrical ends.
- Cylindrical end $\leftrightarrow$ extremality (e.g. maximum J).
- Also, boundary dressed/naked singularity.
- Typical limit of known families: Kerr, Reissner-Nordström.
- More reasonable interest: "Are extremal limits of known families very special?"
- "No, modulo assumptions, families of black holes have extremal limits." (Gabach-Clément, 2010).
- Down to earth interest: "Is there initial data obtainable perturbing known extremal initial data?"
- "Yes, for Kerr" (Dain and Gabach-Clément, 2011).
- What happens with Majumdar-Papapetrou?
- What happens with the evolution of non-stationary extremal initial data?

Field equations

$M = \mathbb{R}^3 \setminus \{0\} \rightarrow$ initial data: $(M, g_{ij}, K_{ij}, E^i, B^i)$

$$R + K^2 - K_{ij}K^{ij} = 2(E_iE^i + B_iB^i)$$

$$D_jK_i^j - D_iK = -2\epsilon_{ijk}E^jB^k = 0$$

$$D_iE^i = 0, \qquad D_iB^i = 0$$

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Hypotheses

- Axial symmetry
  $\mathcal{L}_\eta g_{ij} = 0, \mathcal{L}_\eta K_{ij} = 0, \mathcal{L}_\eta E^i = 0, \mathcal{L}_\eta B^i = 0$
- Time-rotation symmetry
  $\phi \to -\phi \Rightarrow g_{ij} \to g_{ij},\ K_{ij} \to -K_{ij},\ E^i \to E^i,\ B^i \to B^i$
  $\Rightarrow K = 0,\ \int_M |\partial f|_g^2 + Rf^2 d\mu_g > 0$
- Asymptotic structure
  $\rho \to \infty$ asymptotically flat end
  $\rho \to 0$ asymptotically cylindrical end

Field equations

$$g_{ij} = \Phi^4 \tilde{g}_{ij}, \quad K_{ij} = \Phi^{-2} \tilde{K}_{ij}, \quad , E^i = \Phi^{-6} \tilde{E}^i, \quad B^i = \Phi^{-6} \tilde{B}^i$$

$$\tilde{g}_{ij} = e^{2q}(d\rho^2 + dz^2) + \rho^2 d\phi^2$$

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$$\tilde{K}^{ij} = \frac{2}{\rho^2} \tilde{S}^{(i}\eta^{j)}, \quad \tilde{S}^i = \frac{1}{2\rho^2} \tilde{\epsilon}^{ijk} \eta_j \partial_k \omega, \quad \partial_i \chi = F_{ji}\eta^j, \quad \partial_i \psi = *F_{ji}\eta^j$$

$$J = \frac{\omega_- - \omega_+}{8}, \qquad Q_E = \frac{\psi_- - \psi_+}{2}, \qquad Q_B = \frac{\chi_- - \chi_+}{2}$$

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Lichnerowicz equation

$$\Delta \Phi = -\frac{\Delta_2 q}{4} \Phi - \frac{(\partial \omega)^2}{16 \rho^4 \Phi^7} - \frac{(\partial \psi)^2 + (\partial \chi)^2}{4 \rho^2 \Phi^3}$$

$$\Delta = \partial_\rho^2 + \rho^{-1} \partial_\rho + \partial_z^2, \qquad \Delta_2 = \partial_\rho^2 + \partial_z^2$$

Field equations

- Known solution with decay conditions: $(q_0, \omega_0, \psi_0, \chi_0, \Phi_0)$
- Perturbation parameter and perturbation free functions:

$$q_0 \to q_0 + \lambda q, \quad \omega_0 \to \omega_0 + \lambda\omega, \quad \psi_0 \to \psi_0 + \lambda\psi, \quad \chi_0 \to \chi_0 + \lambda\chi$$

- Perturbed solution:

$$\Phi_0 \to \Phi := \Phi_0 + u$$

- Field equation:

$$G(\lambda, u) = 0$$

$$\begin{aligned} G(\lambda, u) \;&=\; \Delta u + \frac{\Delta_2 q_0 u}{4} + \frac{\lambda}{4}\Delta_2 q(\Phi_0 + u) + \frac{(\partial\omega_0 + \lambda\partial\omega)^2}{16\rho^4(\Phi_0 + u)^7} - \frac{(\partial\omega_0)^2}{16\rho^4\Phi_0^7} \\ &\quad + \frac{(\partial\psi_0 + \lambda\partial\psi)^2}{4\rho^2(\Phi_0 + u)^3} - \frac{(\partial\psi_0)^2}{4\rho^2\Phi_0^3} + \frac{(\partial\chi_0 + \lambda\partial\chi)^2}{4\rho^2(\Phi_0 + u)^3} - \frac{(\partial\chi_0)^2}{4\rho^2\Phi_0^3} \end{aligned}$$

Main result

Weighted Sobolev spaces H'^k_δ

$$\|u\|_{L'^2_\delta} = \left[\int_{\mathbb{R}^3\setminus\{0\}} |u|^2 r^{-2\delta-3}\right]^{1/2}, \qquad \|u\|_{H'^k_\delta} = \sum_{j=0}^{k} \|D^j u\|_{L'^2_{\delta-j}}$$

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Theorem

Let $q, \omega, \psi, \chi \in C_0^\infty(\mathbb{R}^3 \setminus \Gamma)$ be arbitrary smooth axially symmetric functions. Then, there is $\lambda_0 > 0$ such that for all $\lambda \in (-\lambda_0, \lambda_0)$ there exists a solution $u(\lambda) \in H'^2_{-1/2}$ of equation $G(\lambda, u) = 0$. The solution $u(\lambda)$ is continuously differentiable in λ and satisfies $\Phi_0 + u(\lambda) > 0$. Moreover, for small λ and small u (in the norm $H'^2_{-1/2}$) the solution $u(\lambda)$ is the unique solution of equation $G(\lambda, u) = 0$.

Steps of the proof

Implicit function theorem

- Choose the Sobolev spaces and show that $\Phi_0 + u > 0$.
- $G : \mathbb{R} \times V \to L'^2_{-5/2}$ is well defined.
- G is continuously differentiable.
- $D_2G(0,0) : H'^2_{-1/2} \to L'^2_{-1/2}$ is an isomorphism.

Conclusions

- Once $u(\lambda)$ is found, we can construct $g_{ij}(\lambda), K_{ij}(\lambda), E^i(\lambda), B^i(\lambda)$.
- This family is differentiable in λ and is close to the initial data.
- The perturbations have the same angular momentum, charges and area of the cylindrical end.
- Several known initial data satisfy the hypotheses: extreme Bowen-York (Dain and Gabach-Clément, 2009), extreme Kerr (Dain and Gabach-Clément, 2011), extreme Reissner-Nordström.
- Estimate on the perturbed conformal factor: $0 < \sqrt{r}\Phi \leq \sqrt{r}\Phi_0 + C(\Phi_0)$.
- $\{t = 0\}$ slice in Majumdar-Papapetrou also satisfies the hypotheses but has several cylindrical ends $\rightarrow$ hopefully in the near future.
- What happens with the evolution? $\rightarrow$ hopefully in the near future, but probably not.