Viability of some classes of static spherically symmetric exact interior solutions as models for compact objec

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Plan

Assumptions

The simplifying framework to solve the Einstein–Maxwell field equations(EMFE) will be:

- To consider a static, and spherically symmetric metric, with electric charge (Since Maxwell's equations reduce to Coulomb's law in this symmetry). Therefore,

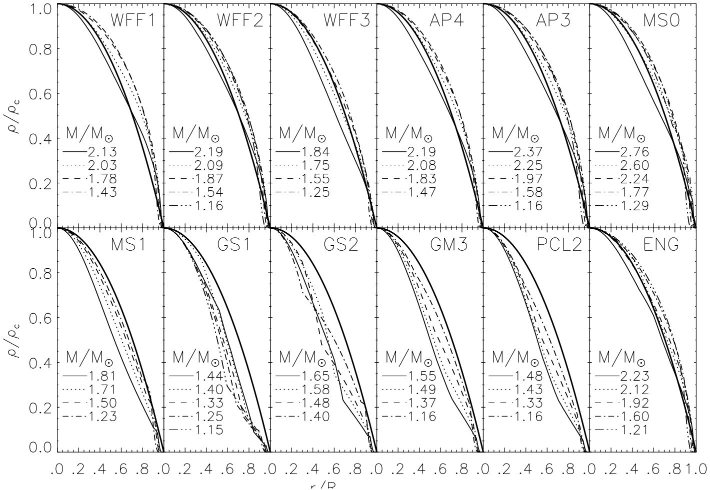
$$\mathrm{d}s^2 = Y^2(r)\,\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{Z(r)} - r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right).$$

- To use a fluid to model the internal "stuff". This fluid may have perfect, or admit anisotropic pressures.
- To match the Schwarzschild Exterior solution at the boundary of uncharged stars, and to use the Reissner-Nordström solution instead for charged ones.

Based on my previous work [1], I chose to extend the known Tolman VII solution.

Why extend Tolman VII

$$\rho = \rho_c \left[1 - \mu \left(\frac{r}{r_b} \right)^2 \right] \implies Z(r) = 1 - br^2 + ar^4 = 1 - \left(\frac{\kappa \rho_c}{3} \right) r^2 + \left(\frac{\kappa \mu \rho_c}{5 r_b^2} \right) r^4$$



The addition of

1. anisotropic pressures is a simple mathematical extension, which is physically motivated since any mixture of perfect fluids *minimally coupled* to each other can be modelled as one anisotropic fluid with energy–momentum tensor [2]:

$$T^i{}_j = \mathrm{diag}\,(\rho, -p_r, -p_\perp, -p_\perp),$$

where $\frac{\Delta}{\kappa} = p_r - p_\perp$, the measure of anisotropy.

2. electric charge is another simple extension, that happens through the addition of Maxwell's equations to the EFE. Physically this allows modelling of young compact objects, resulting in a energy–momentum of

$$T^i{}_j = \mathrm{diag}\left(\frac{q^2}{\kappa r^4}, \frac{q^2}{\kappa r^4}, \frac{-q^2}{\kappa r^4}, \frac{-q^2}{\kappa r^4}\right),$$

where q is a measure of the electric charge.

Charged EMFE with anisotropic pressures

The EMFE with $G = c = 1$, $Y_r \equiv \frac{\mathrm{d}Y}{\mathrm{d}r}$, etc. reduce to the following set :

$$\kappa\rho + \frac{q^2}{r^4} = \frac{1}{r^2} - \frac{Z_r}{r} - \frac{Z}{r^2}, \tag{1}$$

$$\kappa p - \frac{q^2}{r^4} = 2\frac{ZY_r}{rY} + \frac{Z}{r^2} - \frac{1}{r^2}, \tag{2}$$

$$\kappa p_\perp + \frac{q^2}{r^4} = \frac{ZY_{rr}}{Y} + \frac{Z_r Y_r}{2Y} + \frac{ZY_r}{rY} + \frac{Z_r}{2r}, \tag{3}$$

$$\kappa(p+\rho) = 2\frac{ZY_r}{Y} - \frac{Z_r}{r}, \tag{4}$$

$$F_{01} = -\frac{q}{r^2}\frac{Y}{\sqrt{Z}}. \tag{5}$$

These can be solved through a series of coordinate transformations, and substitutions [3], thus,

$$\text{EMFE} \xrightarrow[\Delta=\kappa(p_r-p_\perp)]{\text{define}} \text{A} \xrightarrow[x=r^2]{\text{variable change}} \text{B} \xrightarrow[\xi=\int_0^x \frac{\mathrm{d}\bar{x}}{\sqrt{Z(\bar{x})}}]{\text{variable change}} \text{C} \xrightarrow[Z(x)=1-bx+ax^2]{\text{assume}} \text{D}$$

Algorithm for charged EFE with anisotropic pressures

$$\text{EMFE} \xrightarrow[\Delta=\kappa(p-p_\perp)]{\text{define}} \text{(A)} \xrightarrow[x=r^2]{\text{variable change}} \text{(B)} \xrightarrow[\xi=\int_0^x \frac{\mathrm{d}\bar{x}}{\sqrt{Z(\bar{x})}}]{\text{variable change}} \text{(C)} \xrightarrow[Z(x)=1-bx+ax^2]{\text{assume}} \text{(D)}$$

Each of the stages being given below:

(A):

$$Y_{rr}\{2r^2Z\} + Y_r\{r^2Z_r - 2rZ\} + Y\{rZ_r + 2 + 2r^2\Delta - 4q^2/r^2 - 2Z\} = 0.$$

(B):

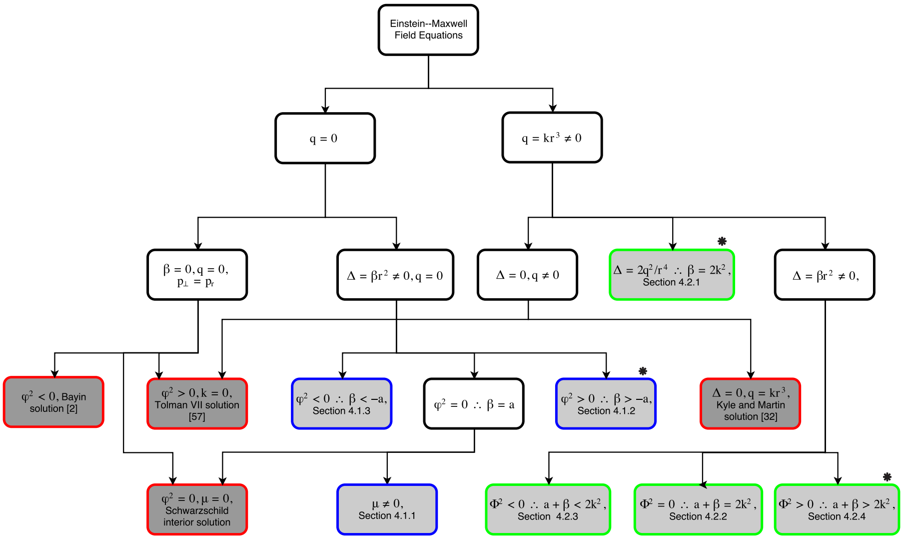
$$Z\{Y_{xx}\} + \frac{Z_xY_x}{2} + \frac{2xZ_x + 2x\Delta + 2 - 2Z - 4q^2/x}{8x^2}Y = 0.$$

(C):

$$Y_{\xi\xi} + \frac{Y}{4}\left(\frac{Z_x}{x} + \frac{\Delta}{x} + \frac{1}{x^2} - \frac{2q^2}{x^3} - \frac{Z}{x^2}\right) = 0.$$

(D):

$$Y_{\xi\xi} + \frac{Y}{4}\left(a + \frac{\Delta}{x} - \frac{2q^2}{x^3}\right) = 0$$



Einstein--Maxwell Field Equations
$q = 0$
$q = kr^3 \neq 0$
$\beta = 0, q = 0,$ $p_\perp = p_r$
$\Delta = \beta r^2 \neq 0, q = 0$
$\Delta = 0, q \neq 0$
$\Delta = 2q^2/r^4 \therefore \beta = 2k^2,$ Section 4.2.1
$\Delta = \beta r^2 \neq 0,$
$\varphi^2 < 0,$ Bayin solution [2]
$\varphi^2 > 0, k = 0,$ Tolman VII solution [57]
$\varphi^2 < 0 \therefore \beta < -a,$ Section 4.1.3
$\varphi^2 = 0 \therefore \beta = a$
$\varphi^2 > 0 \therefore \beta > -a,$ Section 4.1.2
$\Delta = 0, q = kr^3,$ Kyle and Martin solution [32]
$\varphi^2 = 0, \mu = 0,$ Schwarzschild interior solution
$\mu \neq 0,$ Section 4.1.1
$\Phi^2 < 0 \therefore a + \beta < 2k^2,$ Section 4.2.3
$\Phi^2 = 0 \therefore a + \beta = 2k^2,$ Section 4.2.2
$\Phi^2 > 0 \therefore a + \beta > 2k^2,$ Section 4.2.4

Physical constraints

There are additional constraints we can impose on these solutions:

1. The pressure, p_r has to be positive everywhere except at the boundary, where it must vanish. This constraint immediately invalidates some solutions we found.
2. The metric coefficients cannot vanish at any point. This imposes restrictions on the values of a and b for example.
3. The adiabatic speed of sound, $v_s^2 = \frac{\mathrm{d}p}{\mathrm{d}\rho}$ must be finite and causal everywhere. This also restricts the values of a, b, c_1, and c_2.
4. When all the different constraints are taken together, and realistic values for the central density posited, actual measurable values like mass, total charge, radius, and surface density, should be predicted.

The viable solutions that allow for all the constraints and assumptions to be satisfied are the ones that are * starred *

Example of a viable solution : The charged case with anisotropic pressures – varying k

In this solution, the anisotropic parameter β is present as is the charge k, and both can be varied independently. The radial pressure, and speed of sound are shown in the graphs below for fixed values of β.

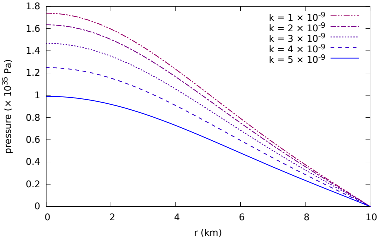


Figure: The radial pressure for $\mu = 0.6, \rho_c = 1 \times 10^{18}\mathrm{kg} \cdot \mathrm{m}^{-3}, r_b = 10\mathrm{km}$

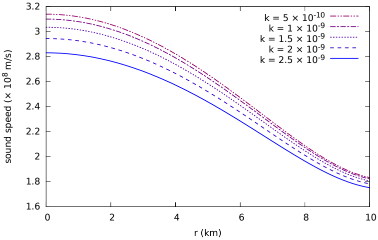


Figure: The speed of sound for $\rho_c = 1 \times 10^{18}\mathrm{kg} \cdot \mathrm{m}^{-3}, r_b = 10\mathrm{km}, \mu = 0.6$

Heuristics for Stability

In GR the differential equations to be perturbed are not simple and are coupled. Thus perturbing them and finding the linear stability of the system is a very lengthy calculation. Over the years a number of heuristics have developed to find out whether a relativistic system will be stable or not.

1. The static stability criterion: predicts all of our new solutions are stable.
2. The (AHN) method [4]: predicts stability if $\beta > 0$.
3. Ponce De Leon's method [5]: predicts that addition of charge stabilizes the star, and addition of anisotropy destabilizes it, but interplay between the two might occur
4. Linear stability analysis: generates the pulsation equation, and some models are stable and some not.

Pulsation equation

$$
\begin{aligned}
&\sigma^2 \, \mathrm{e}^{\lambda_0-\nu_0} \, (p_{r0}+\rho_0) \, \zeta \\
&\quad = \frac{8\pi G}{c^4}(p_{r0}-\Pi_0+\eta_0)(p_{r0}+\rho_0)\zeta \, \mathrm{e}^{\lambda_0} + \frac{2}{r}\delta\Pi - \frac{\zeta(p'_{r0})^2}{p_{r0}+\rho_0} \\
&+ \mathrm{e}^{-(\lambda_0+2\nu_0)/2}\left\{ \mathrm{e}^{(\lambda_0+2\nu_0)/2}\left[\zeta\left(\frac{2\Pi_0}{r}+\frac{4\eta_0}{r}+\eta'_0\right)+\delta\eta\right]\left[\frac{\partial p_{r0}}{\partial \rho_0}-1\right]\right\}' \\
&\qquad - \mathrm{e}^{-(\lambda_0+2\nu_0)/2}\left[\mathrm{e}^{(\lambda_0+3\nu_0)/2}\,\gamma\frac{p_{r0}}{r^2}\left(r^2\zeta\,\mathrm{e}^{-\nu_0/2}\right)'\right]' \\
&\quad + \frac{4p'_{r0}\zeta}{r} + \frac{4\zeta p'_{r0}}{p_{r0}+\rho_0}\left(\frac{\Pi_0}{r}+\frac{4\eta_0}{r}+\eta'_0\right) - \frac{4\zeta}{r}\left(\frac{2\Pi_0}{r}+\frac{4\eta_0}{r}+\eta'_0\right) \\
&\qquad - \frac{\zeta}{p_{r0}+\rho_0}\left(\frac{2\Pi_0}{r}+\frac{4\eta_0}{r}+\eta'_0\right)\left(\frac{2\Pi_0}{r}+\frac{12\eta_0}{r}+3\eta'_0\right). \qquad (6)
\end{aligned}
$$

Allows the determination of the linear stability of general solutions, and the new ones we have found. The signs of the different terms make for difficult predictions.

Applicability and results

The pulsation equation is much more general than our solutions and can be used with any charged solution to the EFE with anisotropic pressures: even those derived from a TOV point of view. When applied to *our* solutions, a summary of the results:

Solution	Parameters	$\sigma_0^2(\mathrm{Hz})$	Stable?
Natural Tolman VII	$\mu = 1$ $\rho_c = 7.43 \times 10^{-10}\mathrm{m}^{-2}$ $r_b = 1 \times 10^4\mathrm{m}$ $k = 0$ $\beta = 0$	54.8	Y
Self-bound Tolman VII	$\mu = 0.7$ $\rho_c = 7.43 \times 10^{-10}\mathrm{m}^{-2}$ $r_b = 1 \times 10^4\mathrm{m}$ $k = 0$ $\beta = 0$	77.6	Y
Tolman VII with anisotropy	$\mu = 1$ $\rho_c = 7.43 \times 10^{-10}\mathrm{m}^{-2}$ $r_b = 1 \times 10^4\mathrm{m}$ $k = 0$ $\beta \sim a/3 = 1 \times 10^{-17}\mathrm{m}^{-4}$	91.4	Y
Tolman VII with anisotropy	$\mu = 1$ $\rho_c = 7.43 \times 10^{-10}\mathrm{m}^{-2}$ $r_b = 1 \times 10^4\mathrm{m}$ $k = 0$ $\beta \sim 10a/3 = 1 \times 10^{-16}\mathrm{m}^{-4}$	821	Y

Solution	Parameters	$\sigma_0^2(\mathrm{Hz})$	Stable?
Self-bound Tolman VII with anisotropy	$\mu = 0.7$ $\rho_c = 7.43 \times 10^{-10}\mathrm{m}^{-2}$ $r_b = 1 \times 10^4\mathrm{m}$ $k = 0$ $\beta \sim a/3 = 1 \times 10^{-17}\mathrm{m}^{-4}$	139	Y
Tolman VII with anisotropy and charge	$\mu = 1$ $\rho_c = 7.43 \times 10^{-10}\mathrm{m}^{-2}$ $r_b = 1 \times 10^4\mathrm{m}$ $k = 1 \times 10^{-10}\mathrm{m}^{-2}$ $\beta \sim a/3 = 1 \times 10^{-17}\mathrm{m}^{-4}$	91.1	Y
Tolman VII with anisotropy and charge[1]	$\mu = 1$ $\rho_c = 7.43 \times 10^{-10}\mathrm{m}^{-2}$ $r_b = 1 \times 10^4\mathrm{m}$ $k = 9 \times 10^{-10}\mathrm{m}^{-2}$ $\beta \sim a/3 = 1 \times 10^{-17}\mathrm{m}^{-4}$	215	Y
Tolman VII with anisotropy and charge[1]	$\mu = 1$ $\rho_c = 7.43 \times 10^{-10}\mathrm{m}^{-2}$ $r_b = 1 \times 10^4\mathrm{m}$ $k = 5 \times 10^{-9}\mathrm{m}^{-2}$ $\beta \sim a/3 = 1 \times 10^{-17}\mathrm{m}^{-4}$	−1360	N

Predictions of masses for causal solutions

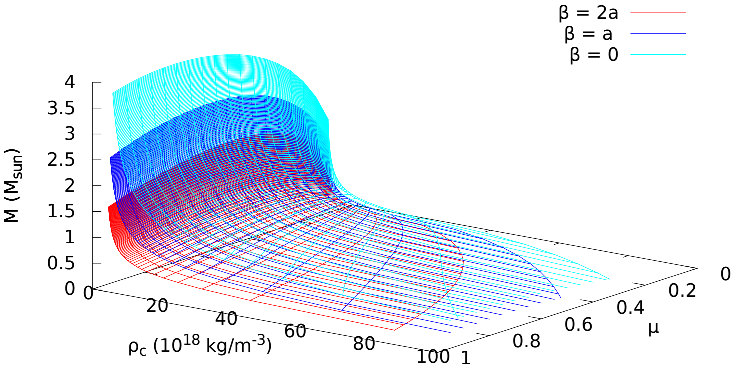


Figure: Anisotropic pressure only, model possibilities.

Predictions of masses for causal solutions

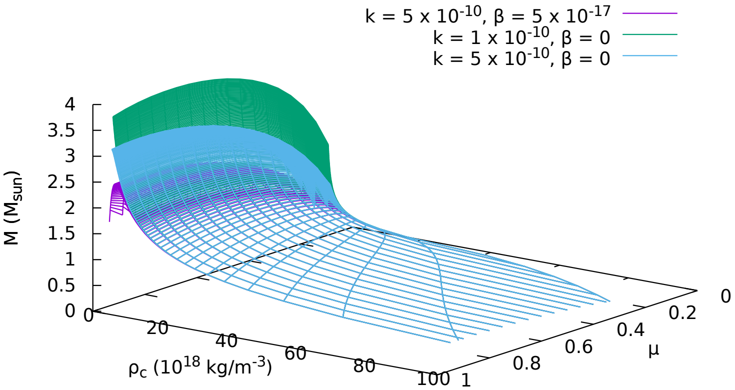
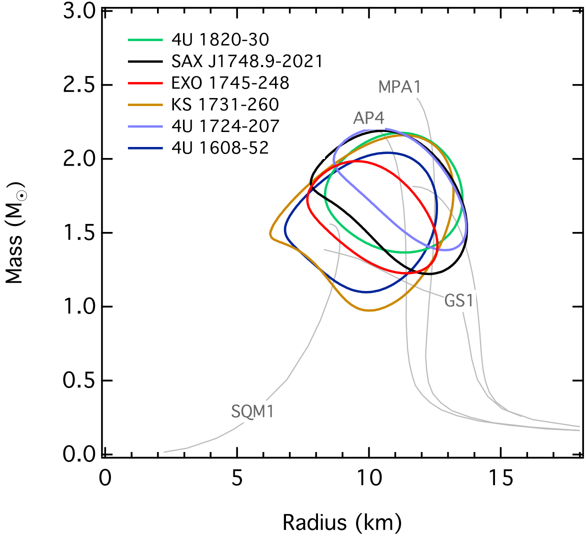


Figure: Anisotropic pressure and charge, model possibilities.

Observations for mass–radius [6]



Conclusion

My project's aim was the following:

> *To find new exact static spherically symmetric solutions for the Einstein–Maxwell system; to prove the linear stability of these solutions; and to use the solutions to model compact objects, while predicting masses and radii of the modelled objects.*

This was tested in the following ways in my thesis:

1. Three physically viable *new* classes of solutions to model compact objects were found.
2. These solutions were proved to produce observables quantities that were compatible with observations.
3. The general linear stability criterion of the class of solution was derived, and proved to hold for the solutions.

This work could be extended in several ways! Ask!

[1] A. M. Raghoonundun and D. W. Hobill, Phys. Rev. D **92**, 124005 (2015).

[2] P. S. Letelier, Phys. Rev. D **22**, 807 (1980).

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[4] H. Abreu, H. Hernández, and L. A. Núñez, Classical and Quantum Gravity **24**, 4631 (2007), 0706.3452.

[5] J. Ponce de Leon, Phys. Rev. D **37**, 309 (1988).

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[7] M. S. R. Delgaty and K. Lake, Computer Physics Communications **115**, 395 (1998).

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Motivation

- A large number of solutions to the interior EFE are known, most of them unphysical. [7] finds that only 9 out of 130 known solutions are physically viable.
- The few that are physical are not well-understood. All their predictions are not known.
- Numerical solutions are difficult to use to generate bounds on physical parameters, or make general statements about classes of solutions. (Use exact solutions)
- Using the TOV equations amounts to using Newton hydrostatics + GR, however with emphasis on the matter content, but does not give the spacetime geometrical structure. (Use full GR, Weyl contributions, and metric functions).
- Current nuclear EOS use a mixture of particles and fields. This cannot be a perfect fluid. [2] (Use imperfect fluids)
- Can we find more physical solutions and are they stable?

Tolman VII's predictions

In 1939, in relation to his solution, Tolman said that,

> *The dependence of p on r, with* $\mathrm{e}^{-\lambda/2}$ *and* $\mathrm{e}^{-\nu}$ *explicitly expressed in terms of r, is so complicated that the solution is not a convenient one for physical considerations [9].*

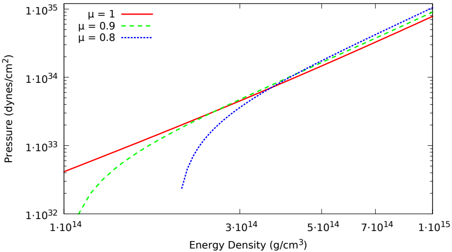


Figure: Pressures of the Tolman VII solution for different self-boundness parameters [1]

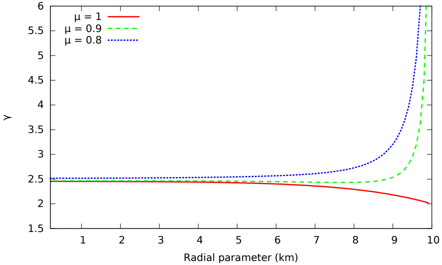


Figure: Adiabatic indices of the Tolman VII EOS, encompassing both natural and self-bound models [1]

Viable solutions I: The Anisotropic case only, $q = k = 0$

In this solution, Only anisotropic pressures, without charge are present. Additionally the anisotropy is such that $\phi^2 > 0$, so that the metric solution for Y in terms of trigonometric functions is possible. The radial pressure, and speed of sound are shown in the graphs below

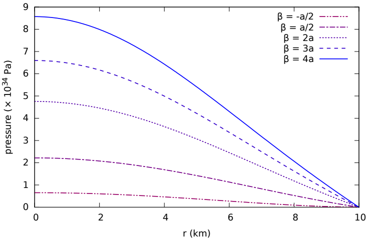


Figure: The radial pressure for $\mu = 0.6, \rho_c = 1 \times 10^{18}\mathrm{kg} \cdot \mathrm{m}^{-3}, r_b = 10\mathrm{km}$

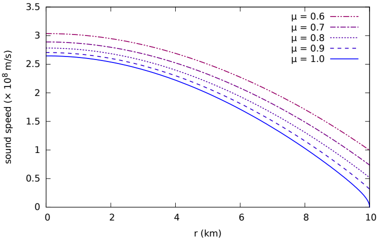
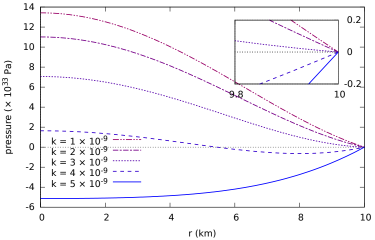


Figure: The speed of sound for $\rho_c = 1 \times 10^{18}\mathrm{kg} \cdot \mathrm{m}^{-3}, r_b = 10\mathrm{km}, \beta = 2a$

Viable solutions II: The anisotropised charge case

In this solution, the anisotropic parameter β is compensated by the charge so that $\beta = 2k^2$. Therefore only one of the parameters β or k is enough to specify the solution. The radial pressure, and speed of sound are shown in the graphs below



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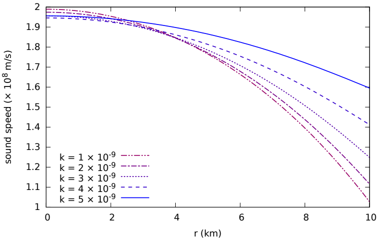
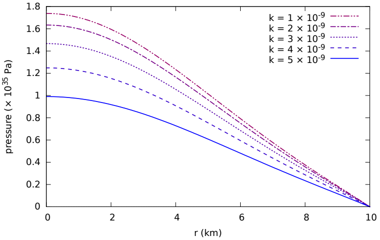


Figure: The speed of sound for $\rho_c = 1 \times 10^{18}\mathrm{kg \cdot m^{-3}}, r_b = 10\mathrm{km}, \mu = 0.6$

Viable solutions III: The charged case with anisotropic pressures – varying k

In this solution, the anisotropic parameter β is present as is the charge k, and both can be varied independently. The radial pressure, and speed of sound are shown in the graphs below for fixed values of β.



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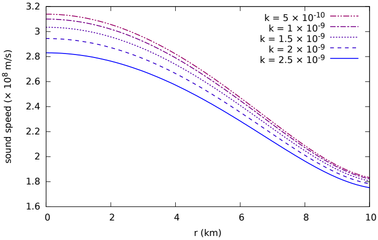

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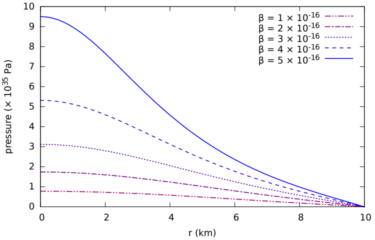

Figure: The radial pressure for $\mu = 1, \rho_c = 1 \times 10^{18}\mathrm{kg} \cdot \mathrm{m}^{-3}, r_b = 10\mathrm{km}, k = 1.2 \times 10^8\mathrm{C} \cdot \mathrm{m}^{-3}$

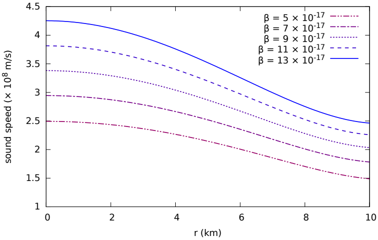

Figure: The speed of sound for $\rho_c = 1 \times 10^{18}\mathrm{kg} \cdot \mathrm{m}^{-3}, r_b = 10\mathrm{km}, \mu = 1, k = 1.2 \times 10^8\mathrm{C} \cdot \mathrm{m}^{-3}$

The Weyl tensor

- The Weyl tensor is the completely antisymmetric part of the Riemann tensor. It is defined through

$$C_{abcd} = R_{abcd} - g_{a[c}R_{d]b} + g_{b[c}R_{d]a} + \frac{R}{3}g_{a[c}g_{d]b}$$

with $R_{ab} = R^{c}{}_{acb}$, and $R = R^{a}{}_{a}$.

- Physically, like Riemann it contains all information about how the shape of a body changes due to geometry and curvature forces.
- Vacuum solutions generate "gravitational force" through Weyl, since $R_{ab} = 0$ in such solutions.
- It is also called the conformal tensor, being invariant if the metric undergoes a conformal transformation.

Birkhoff's theorem

Theorem

- *Any spherically symmetric solution of the vacuum EFE must be static and asymptotically flat in a 4-dimensional manifold.*
- *Any spherically symmetric solution of the Einstein–Maxwell field equations must be stationary and asymptotically flat in a 4-dimensional manifold*

This immediately implies:

- The exterior solution to any spherically symmetric solution must be given by the Schwarzschild metric.
- The exterior geometry of a spherically symmetric charged star must be given by the Reissner-Nordström metric

Buchdahl's inequality

Theorem

For static stars with non-increasing density, the following bound exists:

$$\frac{2m}{r} \leq \frac{8}{9}.$$

The assumptions that go in this inequality are

- the density ρ must not increase with increasing r,
- isotropy of pressures: $p_r = p_\perp \geq 0$,
- if isotropy is relaxed, we then need $p_r \geq p_\perp \geq 0$.

The final aim would be to see which models maximize the compactness parameter $\frac{2m}{r}$