

New conserved currents for vacuum space-times in dimension four with a Killing vector

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Bibliography:

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Outline

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The role of conserved currents

- The energy-momentum conservation principle can be mathematically expressed by means of a **conserved current** and its **conserved charge**.
- A conserved current is a vector field \vec{Z} such that $\nabla_\mu Z^\mu = 0$. Use Gauß Theorem to construct conserved charges.

$$0 = \int_{\Omega} \nabla_\mu Z^\mu = \int_{\mathcal{H}} (n_\mu Z^\mu) d\mathcal{H}, \quad \mathcal{H} \equiv \partial\Omega.$$

If $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2$ then the conserved charges $\mathcal{Q}(\mathcal{H}_1)$, $\mathcal{Q}(\mathcal{H}_2)$ can be defined by

$$\begin{aligned} \mathcal{Q}(\mathcal{H}_1) &\equiv \int_{\mathcal{H}_1} (n_\mu Z^\mu) d\mathcal{H}_1, & \mathcal{Q}(\mathcal{H}_2) &\equiv \int_{\mathcal{H}_2} (n_\mu Z^\mu) d\mathcal{H}_2, \\ \mathcal{Q}(\mathcal{H}_1) &= \mathcal{Q}(\mathcal{H}_2). \end{aligned}$$

- Many times one looks for non-negative (resp. non-positive) conserved charges. This requires finding a conserved current Z^μ and \mathcal{H}_1 , \mathcal{H}_2 with special properties.

The role of conserved currents

Standard examples of conserved currents with non-negative conserved charges:

- Symmetric tensor $T_{\mu\nu}$ fulfilling the **dominant energy condition** and a causal Killing vector ξ^μ . If $\nabla_\mu T^\mu{}_\nu = 0$ and \mathcal{H}_1 or \mathcal{H}_2 are space-like then

$Z^\mu \equiv T^\mu{}_\nu \xi^\nu$ is a conserved current with a non-negative conserved charge.

- Bel-Robinson tensor $B_{\mu\nu\alpha\rho}$ and ξ^μ causal Killing vector field. The vector field

$$Z^\mu \equiv B^\mu{}_{\nu\alpha\rho} \xi^\nu \xi^\alpha \xi^\rho ,$$

is a conserved current in vacuum (use the property $\nabla_\mu B^\mu{}_{\nu\alpha\rho} = 0$) with a non-negative conserved charge if \mathcal{H}_1 or \mathcal{H}_2 are space-like.

The role of conserved currents

Positivity properties of $T_{\mu\nu}$ and $B_{\mu\nu\alpha\rho}$:

- for any $u^\mu, v^\nu, w^\alpha, z^\rho$ causal and future directed one has

$$T_{\mu\nu}u^\mu v^\nu \geq 0, \quad B_{\mu\nu\alpha\rho}u^\mu v^\nu w^\alpha z^\rho \geq 0.$$

- $u^\mu, v^\nu, w^\alpha, z^\rho$ time-like and future-directed

$$T_{\mu\nu}u^\mu v^\nu = 0, \quad B_{\mu\nu\alpha\rho}u^\mu v^\nu w^\alpha z^\rho = 0 \iff T_{\mu\nu} = 0, \quad B_{\mu\nu\alpha\rho} = 0.$$

Due to the charge conservation and the positivity properties, the vanishing of the conserved charges in a spacelike hypersurface Σ entails the vanishing of $T_{\mu\nu}$ or $B_{\mu\nu\alpha\rho}$ in a neighbourhood of $\Sigma \Rightarrow$ the fields defining these tensors also vanish.

Vacuum spacetimes with a Killing vector

Assume that a vacuum space-time $(\mathcal{M}, g_{\mu\nu})$ admits a Killing vector ξ^μ .

Killing condition: $\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$, Killing 2-form: $F_{\mu\nu} \equiv \nabla_{[\mu} \xi_{\nu]} = \nabla_\mu \xi_\nu$.

Self-dual Killing form: $\mathcal{F}_{\mu\nu} \equiv F_{\mu\nu} + i F_{\mu\nu}^*$, $\mathcal{F}^2 \equiv \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu}$.

Ernst 1-form: $\sigma_\nu \equiv 2\xi^\mu \mathcal{F}_{\mu\nu}$.

Self-dual Weyl tensor: $\mathcal{W}_{\mu\nu\lambda\rho} \equiv W_{\mu\nu\lambda\rho} + i W_{\mu\nu\lambda\rho}^*$.

σ_μ closed $\Rightarrow \exists$ a local potential σ (Ernst potential) such that $\sigma_\mu = \nabla_\mu \sigma$.

$\sigma = k + \lambda + 2i\omega$, $k \in \mathbb{C}$, $\lambda \equiv \xi_\mu \xi^\mu$, $\omega \equiv$ twist potential.

The Mars-Simon tensors

Let $\mathcal{I}_{\mu\nu\lambda\rho} \equiv \frac{1}{4} (g_{\mu\lambda}g_{\nu\rho} - g_{\mu\rho}g_{\nu\lambda} + i\eta_{\mu\nu\lambda\rho})$. For any vacuum space-time admitting a Killing vector we define the family of Mars-Simon tensors

$$\mathcal{S}_{\mu\nu\lambda\rho} \equiv \mathcal{W}_{\mu\nu\lambda\rho} + \frac{6}{\sigma} \left(\mathcal{F}_{\mu\nu}\mathcal{F}_{\lambda\rho} - \frac{\mathcal{F}^2}{3}\mathcal{I}_{\mu\nu\lambda\rho} \right), \quad \sigma \neq 0.$$

The Mars-Simon tensors are **Weyl candidates**:

$$\mathcal{S}_{[\mu\nu]\alpha\beta} = \mathcal{S}_{\mu\nu\alpha\beta}, \quad \mathcal{S}_{\mu\nu\alpha\beta} = \mathcal{S}_{\alpha\beta\mu\nu}, \quad \mathcal{S}_{[\mu\nu\alpha]\beta} = 0, \quad \mathcal{S}^{\mu}{}_{\mu\alpha\beta} = 0,$$

and they are self-dual:

$$i \mathcal{S}^*_{\mu\nu\alpha\beta} = \mathcal{S}_{\mu\nu\alpha\beta}.$$

For any point we can choose the Ernst potential in such a way that $\sigma \neq 0$. Therefore the Mars-Simon tensors can be defined (at least locally) for any vacuum space-time.

A local invariant characterisation of the Kerr solution

The Mars-Simon tensor is used in the following important result.

Theorem (Marc Mars (2000))

Let $(\mathcal{M}, g_{\mu\nu})$ be a smooth non-trivial vacuum solution having a Killing vector $\vec{\xi}$ and assume that it fulfills the following conditions

- $(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}) \neq 0$.
- There is a choice of the Ernst potential σ for which

$$\mathcal{S}_{\mu\nu\rho\lambda} = 0, \quad \mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} + \frac{\sigma^4}{4M^2} = 0, \quad M \in \mathbb{R} \setminus \{0\}, \quad \text{Re}(\sigma) - \lambda > 0,$$

- There is at least a point q such that the Killing vector $\vec{\xi}|_q$ does not lie in the 2-plane orthogonal to the 2-plane spanned by the two independent null eigenvectors of $\mathcal{F}_{\mu\nu}|_q$.

Under the previous assumptions the space-time is locally isometric to the Kerr solution.

Our result

Introduce the super-energy tensor of a Mars-Simon tensor:

$$T_{\alpha\beta\gamma\delta} \equiv \mathcal{S}_{\alpha}{}^{\mu\nu}{}_{\beta} \bar{\mathcal{S}}_{\gamma\mu\nu\delta}.$$

$T_{\alpha\beta\gamma\delta}$ has the same positivity and algebraic properties as the Bel-Robinson tensor.

Theorem

For any vacuum solution of the Einstein's field equations $(\mathcal{M}, g_{\mu\nu})$ admitting a Killing vector field, consider an open subset $\mathcal{U} \subset \mathcal{M}$ and a choice of the Ernst potential which is differentiable and non-vanishing. Define the tensor $T_{\alpha\beta\gamma\delta}$ as explained above. Then the current

$$P^{\alpha} \equiv \frac{1}{|\sigma|^6} T^{\alpha}{}_{\beta\gamma\delta} \xi^{\beta} \xi^{\gamma} \xi^{\delta},$$

is conserved in \mathcal{U}

$$\nabla_{\alpha} P^{\alpha} = 0.$$

Remarks

- Use the freedom we have to define the Ernst potential to make a choice which does not vanish at a given point

$$\sigma \rightarrow \sigma + k, \quad k \in \mathbb{C}.$$

Therefore the current \vec{P} can be define at least locally for any vacuum space-time.

- The current \vec{P} is in fact a family which depends on a complex constant k . For example for the Schwarzschild space-time and using Schwarzschild coordinates:

$$\vec{P} = \frac{6M^2|k|^2 r(r-2M)}{|kr-2M|^8} \frac{\partial}{\partial t}, \quad k \in \mathbb{C}.$$

- If the Killing vector $\vec{\xi}$ is causal then the generalised dominant energy property of $T_{\mu\nu\alpha\rho}$ implies that the conserved current \vec{P} is causal too. Moreover under this assumption

$$\vec{P} = 0 \iff T_{\mu\nu\alpha\rho} = 0 \iff \mathcal{S}_{\mu\nu\alpha\rho} = 0.$$

A conserved charge characterising the Kerr solution

Theorem (Kerr conserved charge)

Let $(\mathcal{M}, g_{\mu\nu})$ be a vacuum stationary solution of the Einstein's field equations and assume further that for a given embedded space-like hypersurface $\Sigma \subset \mathcal{M}$, $(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu})|_{\Sigma} \neq 0$ and the Ernst potential is chosen in such a way that it fulfills

$$\left(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} + \frac{\sigma^4}{4M^2} \right) \Big|_{\Sigma} = 0, \quad M \in \mathbb{R} \setminus \{0\}, \quad (\operatorname{Re}(\sigma) - \lambda)|_{\Sigma} > 0.$$

Use the stationary Killing vector and the Ernst potential to define the conserved current \vec{P} . Then the scalar $\mathcal{Q}(\Sigma)$ defined by

$$\mathcal{Q}(\Sigma) \equiv \int_{\Sigma} P^{\mu} n_{\mu} d\Sigma,$$

is non-negative, it vanishes if and only if Σ can be isometrically embedded within an open subset of the Kerr solution and it is a conserved charge.

Open issues

- Attempt to construct similar conserved currents for vacuum with non-vanishing cosmological constant. Use a suitable generalisation of the Mars-Simon tensor family.
- In the definition of the Kerr conserved charge, the hypersurface is spacelike. One could attempt to generalise it for hypersurfaces with a mixed causal character (spacelike-null).
- Render the Kerr conserved charge in terms of initial data for the vacuum Einstein equations or quantities intrinsic to the hypersurface Σ . Use the notion of Killing initial data.