Free hyperboloidal evolution of strong field initial data in spherical symmetry

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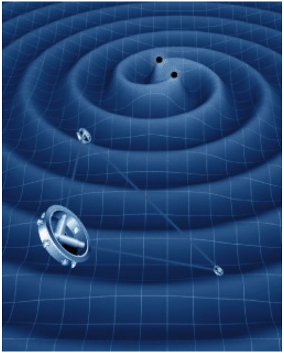
AVV, S. Husa & D. Hilditch, 2014, CQG 32 (2015) 175010, gr-qc/1412.3827.
AVV & S. Husa, ERE2014 proc., JPCS 600 (2015) 1, 012061, gr-qc/1412.4801.
AVV & S. Husa, Marcel Grossmann 14 proc., gr-qc/1601.04079.
AVV & S. Husa, in preparation: gauge conditions & black holes.

Reaching future lightlike infinity



Gravitational waves are only well defined at future null infinity ($\mathscr{I}^+$), where observers of astrophysical events are located.

The study of global properties can also benefit from including $\mathscr{I}^+$.

A possible approach to tackle this problem is to conformally compactify our spacetime: conformally rescale the physical metric $\tilde{g}_{\mu\nu}$

$$g_{\mu\nu} \equiv \Omega^2 \tilde{g}_{\mu\nu}, \tag{1}$$

so that $\Omega|_{\mathscr{I}^+} = 0$ at the appropriate order to keep $g_{\mu\nu}$ finite there.

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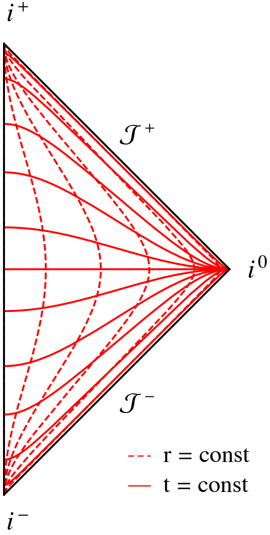

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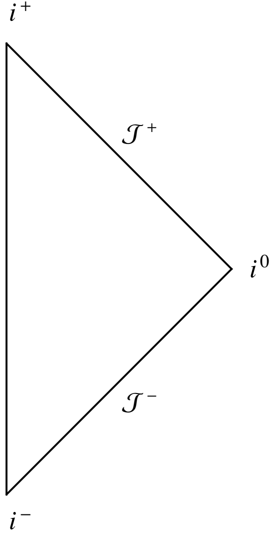
Slicing spacetime



Standard slicing options for the initial value formulation of the Einstein equations, to solve them as an evolution in time:

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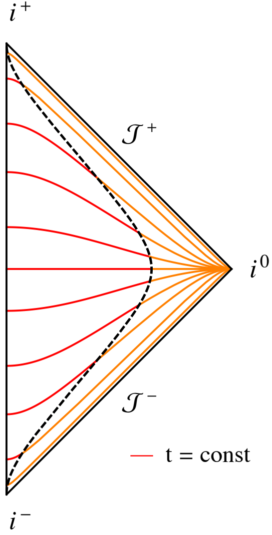
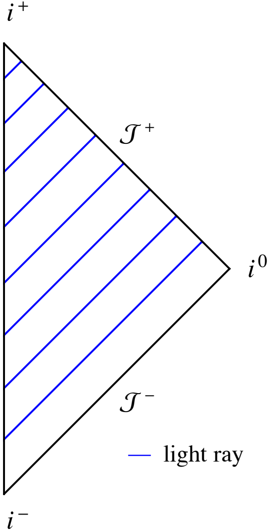


Standard slicing options for the initial value formulation of the Einstein equations, to solve them as an evolution in time:

- Standard Cauchy slices

Slicing spacetime



Standard slicing options for the initial value formulation of the Einstein equations, to solve them as an evolution in time:

- Standard Cauchy slices
- Null slices

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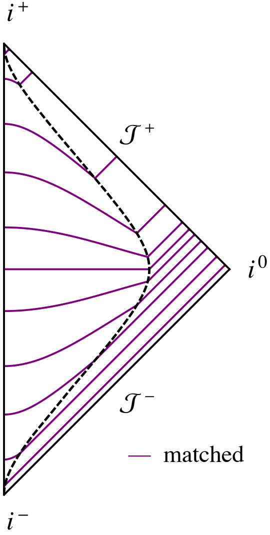
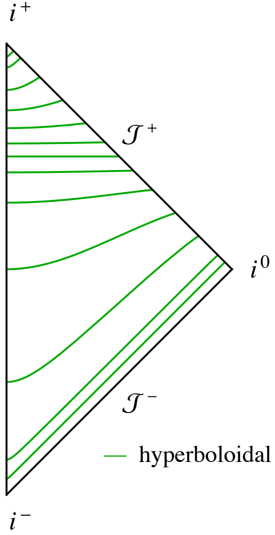


Standard slicing options for the initial value formulation of the Einstein equations, to solve them as an evolution in time:

- Standard Cauchy slices
- Null slices
- Cauchy-Characteristic matching / extraction

Slicing spacetime



Standard slicing options for the initial value formulation of the Einstein equations, to solve them as an evolution in time:

- Standard Cauchy slices
- Null slices
- Cauchy-Characteristic matching / extraction
- Hyperboloidal slices

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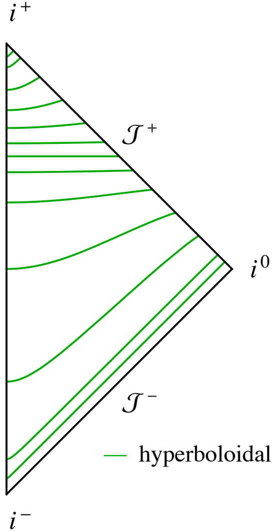

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- Hyperboloidal slices

Advantages of the hyperboloidal approach:

- Extraction at $\mathscr{I}^+$, no approximations.
- Slices spacelike & smooth everywhere.
- More resolution for the central part.

Brief history of the numerical hyperboloidal IVP

- Conformal Field Equations by Friedrich: generality maintained and regularity manifestly shown.
- Numerical implementations by Hübner (tested by Husa, continuum instabilities found) and by Frauendiener.
- Free evolution (generalized harmonic) and a fixed conformal factor by Zenginoğlu: Schwarzschild in spherical symmetry.
- Constrained evolution by Moncrief and Rinne: working axisymmetric code.
- Tetrad formalism by Bardeen, Sarbach and Buchman.

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Main difficulties of the numerical implementation:

- Extra formally divergent terms at $\mathscr{I}^+$ appear in the equations:

$$G_{\mu\nu} = 8\pi\, T_{\mu\nu} - \frac{2}{\Omega}\left(\nabla_\mu\nabla_\nu\Omega - g_{\mu\nu}\nabla^\gamma\nabla_\gamma\Omega\right) - \frac{3}{\Omega^2} g_{\mu\nu}(\nabla_\gamma\Omega)\nabla^\gamma\Omega. \qquad (2)$$

- Non-trivial background ($\tilde{K} \neq 0$), unlike with Cauchy slices.

Basic approach

1. Choose a free evolution to evolve the Einstein equations as an initial value formulation; singularities (punctures, trumpets) are easily avoided, but the system is difficult to stabilize.
2. Use a standard formulation of the Einstein equations: generalized BSSN formulation (GBSSN) or a conformal version of the Z4 formulation (Z4c); different constraint propagation properties - useful to compare.
3. Set a time-independent conformal factor and impose hyperbolic gauge conditions compatible with the non-trivial background: possible to adapt common current gauge choices.
4. Start in spherical symmetry due to its simple numerical implementation. It still includes difficult parts, like the regularization of the radial direction.

Height function and compactification

Hyperboloidal foliations can be described introducing a new time t

$$t = \tilde{t} - h(\tilde{r}) \ , \tag{3}$$

related to the old $\tilde{t}$ by a height function, whose first derivative is

$$h'(\tilde{r}) = -\frac{\frac{K_{CMC}\tilde{r}^3}{3} + C_{CMC}}{\left(1-\frac{2M}{\tilde{r}}\right)\sqrt{\left(\frac{K_{CMC}\tilde{r}^3}{3} + C_{CMC}\right)^2 + \left(1-\frac{2M}{\tilde{r}}\right)\tilde{r}^4}}. \tag{4}$$

To reach $\mathscr{I}^+$ we compactify the radial coordinate into a new r, with $\bar{\Omega}$ determined by imposing a conformally flat spatial metric:

$$\tilde{r} = \frac{r}{\bar{\Omega}}. \tag{5}$$

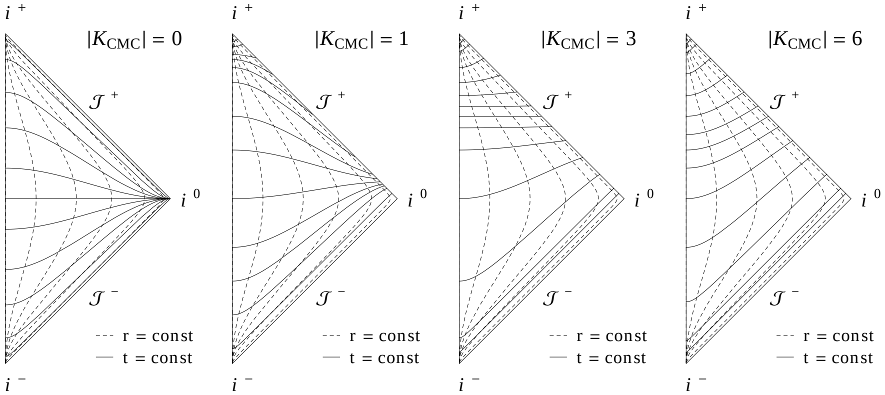
The whole line element is rescaled using the conformal factor

$$\Omega = (-K_{CMC})\frac{r_{\mathscr{I}}^2 - r^2}{6\, r_{\mathscr{I}}}. \tag{6}$$

Flat spacetime on a CMC hyperboloidal foliation

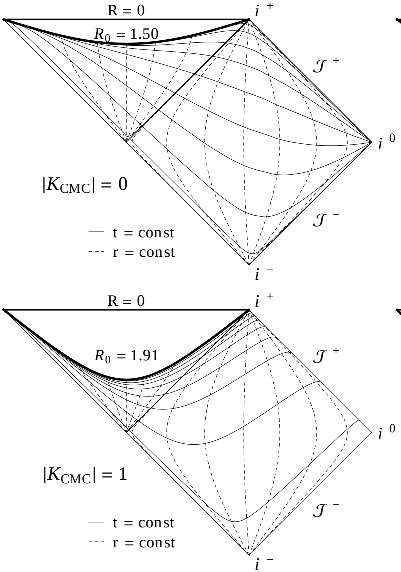
For flat spacetime the height function can be integrated to

$$h(\tilde{r}) = \sqrt{(3/K_{CMC})^2 + \tilde{r}^2}.$$



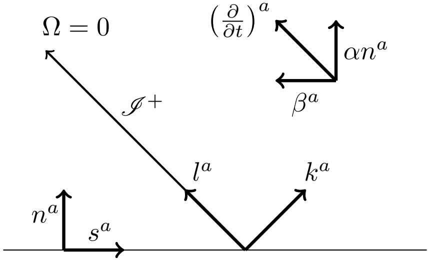
The compactification factor is set to $\bar{\Omega} = \Omega$.

CMC trumpet geometry on a hyperboloidal foliation



Scri-fixing and other coordinate choices

In our spherically symmetric setup we fix the location of $\mathscr{I}^+$ by making the time vector flow along $\mathscr{I}^+$: then $\left(\frac{\partial}{\partial t}\right)^a = \alpha n^a + \beta^a$ is null at $\mathscr{I}^+$.



The conditions are: $\partial_t \Omega|_{\mathscr{I}} = 0$ and $\bar{g}_{tt}|_{\mathscr{I}^+} = -\alpha^2 + \chi^{-1}\gamma_{rr}\beta^{r2}\big|_{\mathscr{I}} = 0$.

The preferred conformal gauge,

$$\Box\Omega|_{\mathscr{I}^+} = 0, \tag{7}$$

holds for some $\dot{\alpha}$ and $\dot{\beta}^r$. It simplifies how the divergent terms cancel & is a requisit for:

The Bondi time at $\mathscr{I}^+$ is related to our code time t via

$$dt_{Bondi} = \frac{\alpha^2\omega}{\beta^r\Omega'}dt, \tag{8}$$

where $\omega = 1$ if the preferred conformal gauge holds.

Hyperbolic slicing and shift conditions

Slicing: Generalized Bona-Massó equation of the form

$$\dot{\alpha} = \beta^r \alpha' - f(\alpha)\,\alpha^2\,(K - K_0) + L_0, \tag{9}$$

with freedom to choose the two functions K_0 and L_0.

- Harmonic: $f(\alpha) = 1$
- 1+log: $f(\alpha) = n_{1+log}/\alpha$
- Constant coefficient: $f(\alpha) = n_{const}/\alpha^2$

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Shift:

- Fixed shift throughout the evolution.
- Gamma-driver shift, with source function and damping:

$$\dot{\beta}^r = \beta^r \beta^{r\,\prime} + \lambda\,\Lambda^r - \eta\,\beta^r + L_0 - \frac{\xi_{\beta^r}}{\Omega}\beta^r. \tag{10}$$

- Harmonic shift, with source function and damping:

$$\dot{\beta}^r = \beta^r \beta^{r\,\prime} + \alpha^2 \chi \Lambda^r + \frac{\alpha^2 \chi'}{2\gamma_{rr}} - \frac{\alpha \chi \alpha'}{\gamma_{rr}} + L_0 - \frac{\xi_{\beta^r}}{\Omega}\beta^r. \tag{11}$$

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BH stationary end state depends on choices like $f(\alpha)$ (and λ).
Matching for physical characteristic gauge speeds at $\mathscr{I}^+$.

Treatment of $\mathscr{I}^+$ and gauges

After experimentation with our implementation:

Required:
- Scri-fixing
- Correct behaviour of gauge variables at $\mathscr{I}^+$ (to account for the non-trivial background) $\rightarrow$ source functions and (if necessary) source terms
- No divergent terms to cause exponential growths

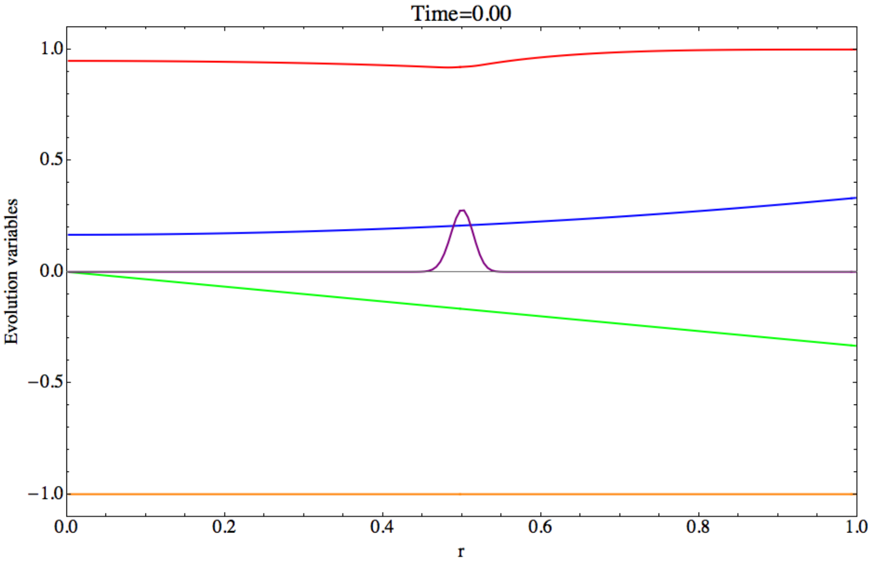
Convenient:
- Preferred conformal gauge
- Bondi time at $\mathscr{I}^+$

Possible:
- 1+log slicing, Gamma-driver shift; matched to the appropriate behaviour at $\mathscr{I}^+$
- Other time parametrizations at $\mathscr{I}^+$

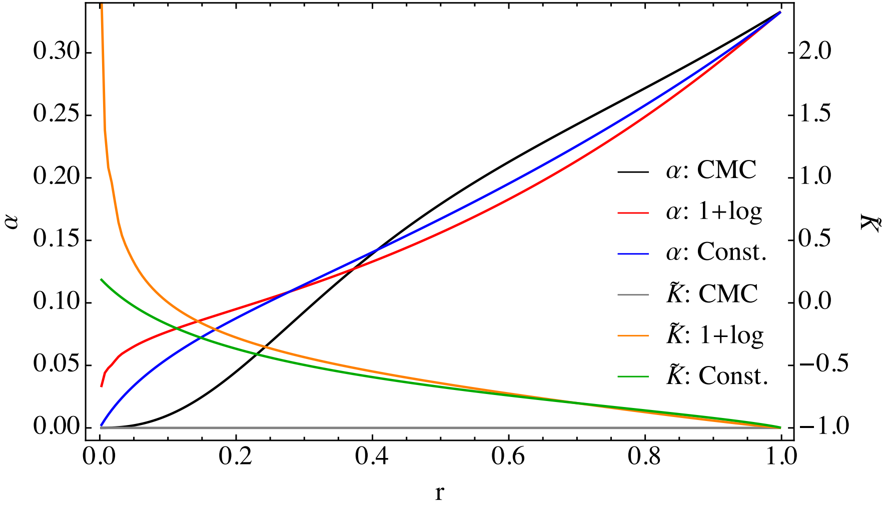
To test:
- Superluminal gauge speeds at $\mathscr{I}^+$

Once necessary conditions satisfied, the implementation is robust: many possible choices!

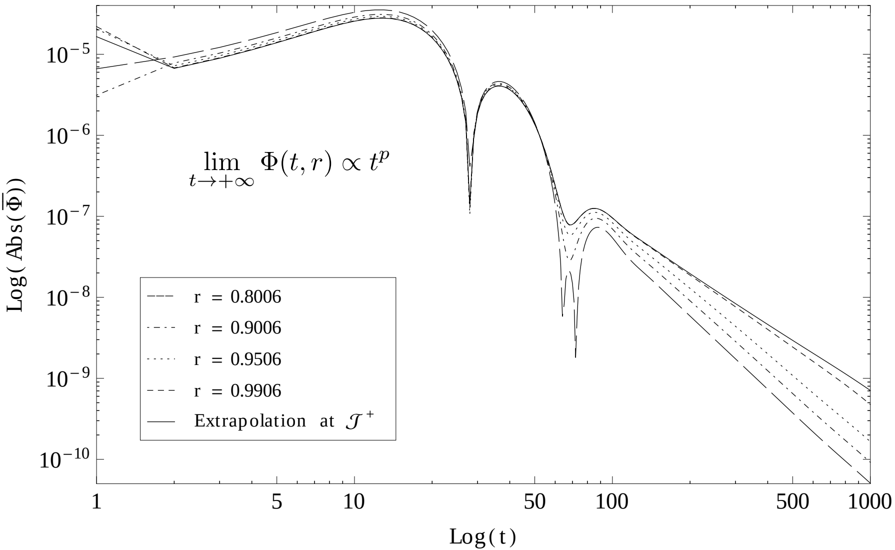
Evolution: χ, $\tilde{K}$, α, β^r, Φ/Ω



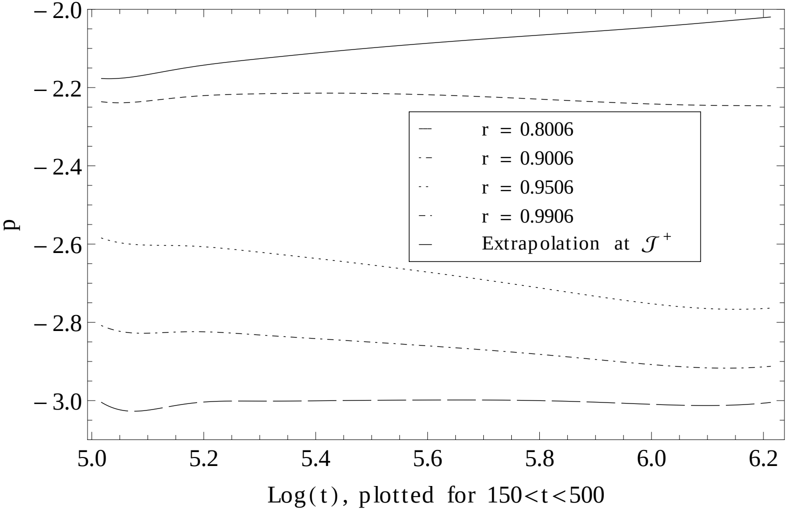
Effect of slicing conditions



Power-law decay tails of the scalar field

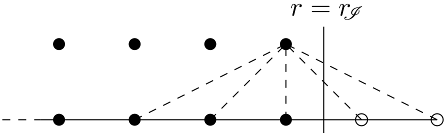


Slopes of the decay tails



Numerical grid at $\mathscr{I}^+$

- Staggered grid:
  simpler implementation; values on $\mathscr{I}^+$ using extrapolation.



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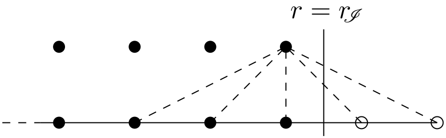
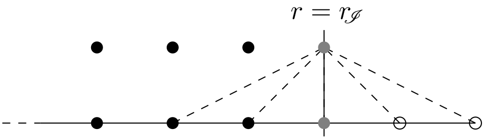
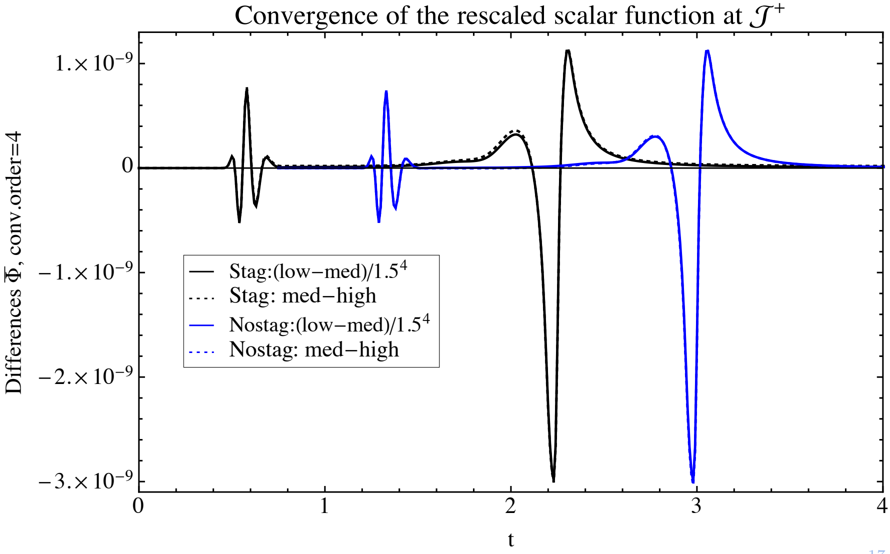


- Non-staggered grid:
  requires regularity conditions at $\mathscr{I}^+$ and calculating the limits of the divergent terms in the equations; quantities given on $\mathscr{I}^+$.



Scalar field - convergence at $\mathscr{I}^+$



Summary

- Our implementation of the hyperboloidal initial value problem in spherical symmetry allows us to evolve a large variety of data. To our knowledge, this is the first stable free evolution achieved with a standard formulation.
- The behaviour of the scalar field signals extracted at $\mathscr{I}^+$ (decay tails, convergence) corresponds to the expected one.
- Compatible gauge conditions have been extensively studied and now we better understand their freedom and robustness.
- The treatment of $\mathscr{I}^+$ has been improved with the introduction of the preferred conformal gauge and the gridpoint on $\mathscr{I}^+$.

We are now ready for further work:

- Simulations in AdS ($\mathscr{I}^+$ is timelike $\rightarrow$ boundary conditions).
- 3-dimensional code and appropriate initial data.

Thank you for your attention!

Questions?

Backup slides

What to do?

To solve the problem the equations need to be regular and a gauge has to be chosen. This can be done in two ways:

Regularize first - Friedrich's conformal field equations approach
: Here one maintains generality and develop a framework where regularity can be shown. New variables are introduced, which leads to a large system of equations. The gauge can be specified afterwards.

Set gauge condition and then regularize
: One can assume "inertial observers" at $\mathscr{I}^+$ and fixed coordinate location for $\mathscr{I}^+$ from the start. The regularization procedures are adapted to the chosen gauge and the final system is much simpler.

Our aim is a robust code for numerical work and thus we will take the second approach.

Implementation of set-gauge-then-regularize

To evolve the Einstein equations as an initial value formulation on a hyperboloidal foliation first setting the gauge and then regularizing, we have mainly two options:

Elliptic hyperbolic problem

- • constrained evolution
- ✗ global equations, hard to avoid singularities
- ✓ easier to stabilize
- ✓ can impose boundary conditions
- ✗ slower
- • Moncrief, Rinne, ...

Purely hyperbolic problem

- • free evolution
- ✓ easier to avoid singularities (using excision or punctures)
- ✗ instabilities
- ✗ cannot impose boundary conditions (have to evolve)
- ✓ faster
- • Zenginoğlu, Bardeen et al., ...

Evolution variables

We use either the Generalized BSSN formulation or a similar conformal version of the Z4 formulation of the Einstein equations.

The line element with our spherically symmetric metric variables is

$$ds^2 = -\alpha^2 dt^2 + \chi^{-1}\left[\gamma_{rr}(dr + \beta^r dt)^2 + \gamma_{\theta\theta} r^2 \left(d\theta^2 + \sin^2\theta d\phi\right)\right] \ . \quad (12)$$

Also the trace of the extrinsic curvatue K, a component A_{rr} of its trace-free part and the contracted difference of Christoffel symbols Λ^r. And for the Z4 formulation its variables Θ and Z_r.

We add a massless scalar field with evolution equation

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\Phi - 2g^{\mu\nu}\nabla_\mu\Phi\frac{\nabla_\nu\Omega}{\Omega} = 0 \ , \quad (13)$$

and its spherically symmetric variables are Φ and $\Pi = \dot{\Phi}$.

GBSSN and Z4 variables

- The three-dimensional conformal metric $\gamma_{ab} = \chi\bar{\gamma}_{ab}$, where $\bar{\gamma}_{ab}$ is the physical metric.
- From the physical extrinsic curvature tensor $\bar{K}_{ab}$:
  - Its trace K.
  - Its conformal trace-free part $A_{ab} = \chi\left(\bar{K}_{ab} - \frac{1}{3}K\right)$.
- The conformal factor $\chi = \left(\frac{\bar{\gamma}}{\gamma}\right)^{\frac{1}{3}}$ (or $\varphi = -\frac{1}{4}\ln\chi$).
- The vector $\Lambda^a \equiv \Delta\Gamma^a = \gamma^{bc}\left(\Gamma^a_{bc} - \hat{\Gamma}^a_{bc}\right)$, where Γ^a_{bc} are the Christoffel symbols calculated from γ_{ab} and $\hat{\Gamma}^a_{bc}$ the ones built from a background metric $\hat{\gamma}_{ab}$.
- The gauge variables:
  - The lapse α.
  - The shift β^a and its auxiliary variable B^a.
- The Z4 variables: Θ and Z_a.

3+1 decomposed equations

$$\partial_\perp \bar\gamma_{ab} = -2\alpha \bar K_{ab} \ , \tag{14a}$$

$$\begin{aligned}\partial_\perp \bar K_{ab} = {} & \alpha\left[R[\bar D]_{ab} - 2\bar K^c_a \bar K_{bc} + \bar K_{ab}(K-2C_{Z4c}\Theta)+2\bar D_{(a}Z_{b)} - \frac{\kappa_1(1+\kappa_2)\bar\gamma_{ab}\Theta}{\Omega}\right] - \bar D_b \bar D_a \alpha \\ & + \frac{3\bar\gamma_{ab}\left[(\partial_\perp\Omega)^2 - \alpha^2 \bar D^c\Omega\bar D_c\Omega\right]}{\alpha\Omega^2} + \frac{4\alpha Z_{(a}\bar D_{b)}\Omega}{\Omega} + \frac{2\alpha\bar D_b\bar D_a\Omega}{\Omega} - \frac{2\alpha\bar\gamma_{ab}Z^c\bar D_c\Omega}{\Omega} \\ & + \frac{\bar\gamma_{ab}\bar D^c\alpha\bar D_c\Omega}{\Omega} + \frac{\alpha\bar\gamma_{ab}\bar\triangle\Omega}{\Omega} + \frac{2\bar K_{ab}\partial_\perp\Omega}{\Omega} + \frac{\bar\gamma_{ab}(K-2C_{Z4c}\Theta)\partial_\perp\Omega}{\Omega} \\ & + \frac{\bar\gamma_{ab}\partial_\perp\alpha\partial_\perp\Omega}{\alpha^2\Omega} - \frac{\bar\gamma_{ab}\partial_\perp\partial_\perp\Omega}{\alpha\Omega} + 4\pi\alpha\left[\bar\gamma_{ab}(S-\rho) - 2S_{ab}\right],\end{aligned} \tag{14b}$$

$$\begin{aligned}\partial_\perp\Theta = {} & \frac{\alpha}{2}\left[R[\bar D] - \bar K_{ab}\bar K^{ab} + K(K-2C_{Z4c}\Theta)+2\bar D_a Z^a - \frac{2\kappa_1(2+\kappa_2)\Theta}{\Omega}\right] - C_{Z4c}Z^a\bar D_a\alpha \\ & + \frac{3\left[(\partial_\perp\Omega)^2 - \alpha^2\bar D^a\Omega\bar D_a\Omega\right]}{\alpha\Omega^2} + \frac{2\alpha\bar\triangle\Omega}{\Omega} + \frac{2(K-2C_{Z4c}\Theta)\partial_\perp\Omega}{\Omega} - 8\pi\alpha\rho \ ,\end{aligned} \tag{15a}$$

$$\begin{aligned}\partial_\perp Z_a = {} & \alpha\left[\bar D_b\bar K_a^b - \bar D_a K + \bar D_a\Theta - 2\bar K_{ab}Z^b - \frac{\kappa_1 Z_a}{\Omega}\right] - C_{Z4c}\Theta\bar D_a\alpha + \frac{2\alpha\Theta\bar D_a\Omega}{\Omega} \\ & - \frac{2\bar D_a\partial_\perp\Omega}{\Omega} - \frac{2\alpha\bar K_a{}^b\bar D_b\Omega}{\Omega} - \frac{2Z_a\partial_\perp\Omega}{\Omega} + \frac{2\bar D_a\alpha\partial_\perp\Omega}{\alpha\Omega} - 8\pi\alpha J_a \ .\end{aligned} \tag{15b}$$

$$\mathcal{H} = R[\bar D] - \bar K_{ab}\bar K^{ab} + K^2 + \frac{6\left[(\partial_\perp\Omega)^2 - \alpha^2\bar D^a\Omega\bar D_a\Omega\right]}{\alpha^2\Omega^2} + \frac{4\bar\triangle\Omega}{\Omega} + \frac{4K\partial_\perp\Omega}{\alpha\Omega} - 16\pi\rho \ , \tag{16a}$$

$$\mathcal{M}^a = \bar D_b\bar K^{ab} - \bar\gamma^{ab}\bar D_b K - \frac{2\bar K^{ab}\bar D_b\Omega}{\Omega} - \frac{2\bar\gamma^{ab}\bar D_b\partial_\perp\Omega}{\alpha\Omega} + \frac{2\bar\gamma^{ab}\bar D_b\alpha\partial_\perp\Omega}{\alpha^2\Omega} - 8\pi J^a \ . \tag{16b}$$

GBSSN and conformal Z4 equations I

$$\partial_\perp \chi = \frac{2}{3}\alpha\chi(K+2\Theta) + \frac{1}{3}\chi\partial_\perp \ln\gamma \ , \qquad \partial_\perp \gamma_{ab} = -2A_{ab}\alpha + \frac{1}{3}\gamma_{ab}\partial_\perp \ln\gamma \ , \tag{17a}$$

$$\begin{aligned}\partial_\perp A_{ab} = & \Big[\alpha\chi\left(R[D]_{ab} + 2D_{(a}Z_{b)}\right) - \chi D_a D_b\alpha - D_{(a}\alpha D_{b)}\chi - \frac{\alpha D_a\chi D_b\chi}{4\chi} + \frac{1}{2}\alpha D_a D_b\chi \\ & +2Z_{(a}\alpha D_{b)}\chi + \frac{2\alpha D_{(a}\chi D_{b)}\Omega}{\Omega} + \frac{2\alpha\chi D_a D_b\Omega}{\Omega} + \frac{4\alpha\chi Z_{(a}D_{b)}\Omega}{\Omega} - 8\pi\alpha\chi S_{ab}\Big]^{TF} \\ & -2\alpha A^c_a A_{bc} + \alpha A_{ab}[K+2(1-C_{Z4c})\Theta] + \frac{2A_{ab}\partial_\perp\Omega}{\Omega} + \frac{1}{3}A_{ab}\partial_\perp \ln\gamma \ ,\end{aligned} \tag{17b}$$

$$\begin{aligned}\partial_\perp K = & \ \alpha\left[A_{ab}A^{ab} + \frac{1}{3}(K+2\Theta)^2 + \frac{\kappa_1(1-\kappa_2)\Theta}{\Omega}\right] - \chi\triangle\alpha + \frac{1}{2}D^a\alpha D_a\chi + 2C_{Z4c}Z^a D_a\alpha \\ & + \frac{3[(\partial_\perp\Omega)^2 - \alpha^2\chi D^a\Omega D_a\Omega]}{\Omega^2\alpha} - \frac{2\alpha Z^a D_a\Omega}{\Omega} + \frac{3\chi D^a\alpha D_a\Omega}{\Omega} - \frac{\alpha D^a\chi D_a\Omega}{2\Omega} + \frac{\alpha\chi\triangle\Omega}{\Omega} \\ & + \frac{[K+2(1+C_{Z4c})\Theta]\partial_\perp\Omega}{\Omega} + \frac{3\partial_\perp\alpha\partial_\perp\Omega}{\Omega\alpha^2} - \frac{3\partial_\perp\partial_\perp\Omega}{\Omega\alpha} + 4\pi\alpha(\rho+S) \ ,\end{aligned} \tag{17c}$$

$$\begin{aligned}\partial_\perp \Lambda^a = & \ \mathcal{C}^b(D_b\beta^a) + \alpha\left[2A^{bc}\Delta\Gamma^a_{bc} - \frac{4}{3}D^a K - \frac{2}{3}D^a\Theta - \frac{3A^{ab}D_b\chi}{\chi} - \frac{4Z^a(K+2\Theta)}{3\chi} - \frac{2\kappa_1 Z^a}{\Omega\chi}\right] \\ & +\gamma^{bc}\hat{D}_b\hat{D}_c\beta^a - \gamma^{bc}R[\hat{D}]^a{}_{bcd}\beta^d - 2A^{ab}D_b\alpha - 2C_{Z4c}\Theta D^a\alpha - \frac{4\alpha A^{ab}D_b\Omega}{\Omega} \\ & - \frac{4\alpha(K-\Theta)D^a\Omega}{3\Omega} - \frac{4D^a\partial_\perp\Omega}{\Omega} + \frac{4D^a\alpha\partial_\perp\Omega}{\Omega\alpha} - \frac{4Z^a\partial_\perp\Omega}{\Omega\chi} \\ & - \frac{1}{6}D^a\partial_\perp \ln\gamma - \frac{1}{3}\Delta\Gamma^a\partial_\perp \ln\gamma - \frac{2Z^a\partial_\perp \ln\gamma}{3\chi} - \frac{16\pi J^a\alpha}{\chi} \ ,\end{aligned} \tag{17d}$$

GBSSN and conformal Z4 equations II

$$\partial_\perp \Theta = \frac{\alpha}{2}\left[\chi(R[D] + 2D^a Z_a) - A_{ab}A^{ab} + \frac{2}{3}(K+2\Theta)^2 - 2C_{Z4c}\Theta(K+2\Theta) - \frac{2\kappa_1(2+\kappa_2)\Theta}{\Omega}\right]$$
$$+\alpha \triangle \chi - \frac{5\alpha D^a\chi D_a\chi}{4\chi} - C_{Z4c}Z^a D_a\alpha - \frac{\alpha Z^a D_a\chi}{2\chi} + \frac{2\alpha\chi \triangle \Omega}{\Omega} - \frac{\alpha D^a\chi D_a\Omega}{\Omega}$$
$$+\frac{3[(\partial_\perp\Omega)^2 - \alpha^2\chi D^a\Omega D_a\Omega]}{\Omega^2\alpha} + \frac{2[K+2(1-C_{Z4c})\Theta]\partial_\perp\Omega}{\Omega} - 8\pi\alpha\rho \ . \tag{18a}$$

$$\mathcal{H} = \chi R[D] - A_{ab}A^{ab} + \frac{2}{3}(K+2\Theta)^2 + 2\triangle\chi - \frac{5D^a\chi D_a\chi}{2\chi} + \frac{6(\partial_\perp\Omega)^2}{\Omega^2\alpha^2}$$
$$-\frac{6\chi D^a\Omega D_a\Omega}{\Omega^2} - \frac{2D^a\chi D_a\Omega}{\Omega} + \frac{4\chi\triangle\Omega}{\Omega} + \frac{4(K+2\Theta)\partial_\perp\Omega}{\Omega\alpha} - 16\pi\rho \ , \tag{19a}$$

$$\mathcal{M}_a = D_b A_a^b - \frac{2}{3}D_a(K+2\Theta) - \frac{3A_a^b D_b\chi}{2\chi} - \frac{2A_a^b D_b\Omega}{\Omega} - \frac{2(K+2\Theta)D_a\Omega}{3\Omega} - \frac{2D_a\partial_\perp\Omega}{\Omega\alpha}$$
$$+\frac{2D_a\alpha\partial_\perp\Omega}{\Omega\alpha^2} - 8\pi J_a \ , \tag{19b}$$

$$\mathcal{C}^a = \Lambda^a - \Delta\Gamma^a - \frac{2Z^a}{\chi} \ . \tag{19c}$$

Spherically symmetric equations

$$\dot{\chi} = \beta^r\chi' + \frac{2\alpha\chi(K+2\Theta)}{3} - \frac{\beta^r\gamma_{rr}'\chi}{3\gamma_{rr}} - \frac{2\beta^r\gamma_{\theta\theta}'\chi}{3\gamma_{\theta\theta}} - \frac{2\beta^{r\prime}\chi}{3} - \frac{4\beta^r\chi}{3r} + \frac{2\beta^r\chi\Omega'}{\Omega} - \frac{2\alpha\chi}{a\Omega}\ , \quad (20a)$$

Spherically symmetric equations

$$\dot{\chi} = \beta^r \chi' + \frac{2\alpha\chi(K+2\Theta)}{3} - \frac{\beta^r \gamma_{rr}' \chi}{3\gamma_{rr}} - \frac{2\beta^r \gamma_{\theta\theta}' \chi}{3\gamma_{\theta\theta}} - \frac{2\beta^{r\prime}\chi}{3} - \frac{4\beta^r\chi}{3r} + \frac{2\beta^r\chi\Omega'}{\Omega} - \frac{2\alpha\chi}{a\Omega}, \quad (20a)$$

$$\dot{\gamma_{rr}} = -2A_{rr}\alpha + \frac{2\beta^r\gamma_{rr}'}{3} - \frac{2\gamma_{rr}\beta^r\gamma_{\theta\theta}'}{3\gamma_{\theta\theta}} + \frac{4\gamma_{rr}\beta^{r\prime}}{3} - \frac{4\gamma_{rr}\beta^r}{3r}, \quad (20b)$$

Spherically symmetric equations

$$\dot{\chi} = \beta^r\chi' + \frac{2\alpha\chi(K+2\Theta)}{3} - \frac{\beta^r\gamma_{rr}'\chi}{3\gamma_{rr}} - \frac{2\beta^r\gamma_{\theta\theta}'\chi}{3\gamma_{\theta\theta}} - \frac{2\beta^{r\prime}\chi}{3} - \frac{4\beta^r\chi}{3r} + \frac{2\beta^r\chi\Omega'}{\Omega} - \frac{2\alpha\chi}{a\Omega}, \quad (20a)$$

$$\dot{\gamma_{rr}} = -2A_{rr}\alpha + \frac{2\beta^r\gamma_{rr}'}{3} - \frac{2\gamma_{rr}\beta^r\gamma_{\theta\theta}'}{3\gamma_{\theta\theta}} + \frac{4\gamma_{rr}\beta^{r\prime}}{3} - \frac{4\gamma_{rr}\beta^r}{3r}, \quad (20b)$$

$$\dot{\gamma_{\theta\theta}} = \frac{A_{rr}\gamma_{\theta\theta}\alpha}{\gamma_{rr}} - \frac{\gamma_{\theta\theta}\beta^r\gamma_{rr}'}{3\gamma_{rr}} + \frac{\beta^r\gamma_{\theta\theta}'}{3} - \frac{2\gamma_{\theta\theta}\beta^{r\prime}}{3} + \frac{2\gamma_{\theta\theta}\beta^r}{3r}, \quad (20c)$$

Spherically symmetric equations

$$\dot{\chi} = \beta^r\chi' + \frac{2\alpha\chi(K+2\Theta)}{3} - \frac{\beta^r\gamma_{rr}'\chi}{3\gamma_{rr}} - \frac{2\beta^r\gamma_{\theta\theta}'\chi}{3\gamma_{\theta\theta}} - \frac{2\beta^{r\prime}\chi}{3} - \frac{4\beta^r\chi}{3r} + \frac{2\beta^r\chi\Omega'}{\Omega} - \frac{2\alpha\chi}{a\Omega}\ , \tag{20a}$$

$$\dot{\gamma_{rr}} = -2A_{rr}\alpha + \frac{2\beta^r\gamma_{rr}'}{3} - \frac{2\gamma_{rr}\beta^r\gamma_{\theta\theta}'}{3\gamma_{\theta\theta}} + \frac{4\gamma_{rr}\beta^{r\prime}}{3} - \frac{4\gamma_{rr}\beta^r}{3r}\ , \tag{20b}$$

$$\dot{\gamma_{\theta\theta}} = \frac{A_{rr}\gamma_{\theta\theta}\alpha}{\gamma_{rr}} - \frac{\gamma_{\theta\theta}\beta^r\gamma_{rr}'}{3\gamma_{rr}} + \frac{\beta^r\gamma_{\theta\theta}'}{3} - \frac{2\gamma_{\theta\theta}\beta^{r\prime}}{3} + \frac{2\gamma_{\theta\theta}\beta^r}{3r}\ , \tag{20c}$$

$$\begin{aligned}\dot{A_{rr}} = {} & \beta^r A_{rr}' + \frac{2}{3}\gamma_{rr}\alpha\chi\Lambda^{r\prime} - \frac{\alpha\chi\gamma_{rr}''}{3\gamma_{rr}} + \frac{\alpha\chi\gamma_{\theta\theta}''}{3\gamma_{\theta\theta}} - \frac{2\chi\alpha''}{3} + \frac{\alpha\chi''}{3} + \alpha A_{rr}\left[K + 2(1-C_{Z4c})\Theta\right] \\ & - \frac{2\alpha A_{rr}^2}{\gamma_{rr}} + \frac{4\beta^{r\prime}A_{rr}}{3} - \frac{4\beta^r A_{rr}}{3r} + \frac{\alpha\chi\left(\gamma_{rr}'\right)^2}{2\gamma_{rr}^2} - \frac{2\alpha\chi\left(\gamma_{\theta\theta}'\right)^2}{3\gamma_{\theta\theta}^2} - \frac{\alpha\left(\chi'\right)^2}{6\chi} - \frac{2\gamma_{rr}\alpha\Lambda^r\chi}{3r} \\ & + \frac{2\alpha\chi}{r^2}\left(\frac{\gamma_{rr}}{\gamma_{\theta\theta}} - 1\right) - \frac{A_{rr}\beta^r\gamma_{rr}'}{3\gamma_{rr}} - \frac{2A_{rr}\beta^r\gamma_{\theta\theta}'}{3\gamma_{\theta\theta}} + \frac{\alpha\Lambda^r\chi\gamma_{rr}'}{3} - \frac{\alpha\Lambda^r\chi\gamma_{rr}\gamma_{\theta\theta}'}{3\gamma_{\theta\theta}} - \frac{2\alpha\chi\gamma_{rr}'}{3\gamma_{\theta\theta}r} \\ & + \frac{2\alpha\chi\gamma_{rr}\gamma_{\theta\theta}'}{\gamma_{\theta\theta}^2 r} - \frac{4\alpha\chi\gamma_{\theta\theta}'}{3\gamma_{\theta\theta}r} + \frac{\chi\alpha'\gamma_{rr}'}{3\gamma_{rr}} + \frac{\chi\alpha'\gamma_{\theta\theta}'}{3\gamma_{\theta\theta}} + \frac{2\chi\alpha'}{3r} - \frac{2\alpha'\chi'}{3} - \frac{\alpha\chi'\gamma_{rr}'}{6\gamma_{rr}} - \frac{\alpha\chi'\gamma_{\theta\theta}'}{6\gamma_{\theta\theta}} \\ & - \frac{\alpha\chi'}{3r} + \frac{4}{3}Z_r\alpha\chi' - \frac{2\alpha\chi\gamma_{rr}'\Omega'}{3\gamma_{rr}\Omega} - \frac{2\alpha\chi\gamma_{\theta\theta}'\Omega'}{3\gamma_{\theta\theta}\Omega} + \frac{4\alpha\chi'\Omega'}{3\Omega} + \frac{A_{rr}\beta^r\Omega'}{\Omega} - \frac{3\alpha A_{rr}}{a\Omega} \\ & + \frac{4\alpha\chi\Omega''}{3\Omega} - \frac{4\alpha\chi\Omega'}{3r\Omega} + \frac{8Z_r\alpha\chi\Omega'}{3\Omega} - \frac{16}{3}\pi\alpha\chi\left(\Phi'\right)^2\ ,\end{aligned} \tag{20d}$$

Spherically symmetric equations

$$\dot{\chi} = \beta^r\chi' + \frac{2\alpha\chi(K+2\Theta)}{3} - \frac{\beta^r\gamma_{rr}'\chi}{3\gamma_{rr}} - \frac{2\beta^r\gamma_{\theta\theta}'\chi}{3\gamma_{\theta\theta}} - \frac{2\beta^{r\prime}\chi}{3} - \frac{4\beta^r\chi}{3r} + \frac{2\beta^r\chi\Omega'}{\Omega} - \frac{2\alpha\chi}{a\Omega}, \quad (20a)$$

$$\dot{\gamma_{rr}} = -2A_{rr}\alpha + \frac{2\beta^r\gamma_{rr}'}{3} - \frac{2\gamma_{rr}\beta^r\gamma_{\theta\theta}'}{3\gamma_{\theta\theta}} + \frac{4\gamma_{rr}\beta^{r\prime}}{3} - \frac{4\gamma_{rr}\beta^r}{3r}, \quad (20b)$$

$$\dot{\gamma_{\theta\theta}} = \frac{A_{rr}\gamma_{\theta\theta}\alpha}{\gamma_{rr}} - \frac{\gamma_{\theta\theta}\beta^r\gamma_{rr}'}{3\gamma_{rr}} + \frac{\beta^r\gamma_{\theta\theta}'}{3} - \frac{2\gamma_{\theta\theta}\beta^{r\prime}}{3} + \frac{2\gamma_{\theta\theta}\beta^r}{3r}, \quad (20c)$$

$$\begin{aligned}\dot{A_{rr}} = {} & \beta^r A_{rr}' + \frac{2}{3}\gamma_{rr}\alpha\chi\Lambda^{r\prime} - \frac{\alpha\chi\gamma_{rr}''}{3\gamma_{rr}} + \frac{\alpha\chi\gamma_{\theta\theta}''}{3\gamma_{\theta\theta}} - \frac{2\chi\alpha''}{3} + \frac{\alpha\chi''}{3} + \alpha A_{rr}\left[K+2(1-C_{Z4c})\Theta\right] \\
& - \frac{2\alpha A_{rr}^2}{\gamma_{rr}} + \frac{4\beta^{r\prime}A_{rr}}{3} - \frac{4\beta^r A_{rr}}{3r} + \frac{\alpha\chi\left(\gamma_{rr}'\right)^2}{2\gamma_{rr}^2} - \frac{2\alpha\chi\left(\gamma_{\theta\theta}'\right)^2}{3\gamma_{\theta\theta}^2} - \frac{\alpha\left(\chi'\right)^2}{6\chi} - \frac{2\gamma_{rr}\alpha\Lambda^r\chi}{3r} \\
& + \frac{2\alpha\chi}{r^2}\left(\frac{\gamma_{rr}}{\gamma_{\theta\theta}} - 1\right) - \frac{A_{rr}\beta^r\gamma_{rr}'}{3\gamma_{rr}} - \frac{2A_{rr}\beta^r\gamma_{\theta\theta}'}{3\gamma_{\theta\theta}} + \frac{\alpha\Lambda^r\chi\gamma_{rr}'}{3} - \frac{\alpha\Lambda^r\chi\gamma_{rr}\gamma_{\theta\theta}'}{3\gamma_{\theta\theta}} - \frac{2\alpha\chi\gamma_{rr}'}{3\gamma_{\theta\theta}r} \\
& + \frac{2\alpha\chi\gamma_{rr}\gamma_{\theta\theta}'}{\gamma_{\theta\theta}^2 r} - \frac{4\alpha\chi\gamma_{\theta\theta}'}{3\gamma_{\theta\theta}r} + \frac{\chi\alpha'\gamma_{rr}'}{3\gamma_{rr}} + \frac{\chi\alpha'\gamma_{\theta\theta}'}{3\gamma_{\theta\theta}} + \frac{2\chi\alpha'}{3r} - \frac{2\alpha'\chi'}{3} - \frac{\alpha\chi'\gamma_{rr}'}{6\gamma_{rr}} - \frac{\alpha\chi'\gamma_{\theta\theta}'}{6\gamma_{\theta\theta}} \\
& - \frac{\alpha\chi'}{3r} + \frac{4}{3}Z_r\alpha\chi' - \frac{2\alpha\chi\gamma_{rr}'\Omega'}{3\gamma_{rr}\Omega} - \frac{2\alpha\chi\gamma_{\theta\theta}'\Omega'}{3\gamma_{\theta\theta}\Omega} + \frac{4\alpha\chi'\Omega'}{3\Omega} + \frac{A_{rr}\beta^r\Omega'}{\Omega} - \frac{3\alpha A_{rr}}{a\Omega} \\
& + \frac{4\alpha\chi\Omega''}{3\Omega} - \frac{4\alpha\chi\Omega'}{3r\Omega} + \frac{8Z_r\alpha\chi\Omega'}{3\Omega} - \frac{16}{3}\pi\alpha\chi\left(\Phi'\right)^2,\end{aligned} \quad (20d)$$

Spherically symmetric equations

$$\dot{\chi} = \text{Principal part} + \ldots + \text{Z4 terms} + \frac{2\beta^r\chi\Omega'}{\Omega} - \frac{2\alpha\chi}{a\Omega}\ , \tag{20a}$$

$$\dot{\gamma_{rr}} = \text{Principal part} + \ldots\ , \tag{20b}$$

$$\dot{\gamma_{\theta\theta}} = \text{Principal part} + \ldots\ , \tag{20c}$$

$$\dot{A_{rr}} = \text{Principal part} + \ldots + \text{Z4 terms} + \frac{A_{rr}\beta^r\Omega'}{\Omega} - \frac{3\alpha A_{rr}}{a\Omega} + \Omega\ \text{terms} + \text{Matter terms}\ , \tag{20d}$$

Spherically symmetric equations

$$\dot{\chi} = \text{Principal part} + \ldots + \text{Z4 terms} + \frac{2\beta^r\chi\Omega'}{\Omega} - \frac{2\alpha\chi}{a\Omega} \,, \tag{20a}$$

$$\dot{\gamma_{rr}} = \text{Principal part} + \ldots \,, \tag{20b}$$

$$\dot{\gamma_{\theta\theta}} = \text{Principal part} + \ldots \,, \tag{20c}$$

$$\dot{A_{rr}} = \text{Principal part} + \ldots + \text{Z4 terms} + \frac{A_{rr}\beta^r\Omega'}{\Omega} - \frac{3\alpha A_{rr}}{a\Omega} + \Omega \text{ terms} + \text{Matter terms} \,, \tag{20d}$$

$$\begin{aligned}\dot{K} = {} & \beta^r K' - \frac{\chi\alpha''}{\gamma_{rr}} + \frac{\alpha}{3}(K+2\Theta)^2 + \frac{3\alpha A_{rr}^2}{2\gamma_{rr}^2} + \frac{\alpha'\gamma_{rr}'\chi}{2\gamma_{rr}^2} - \frac{\alpha'\gamma_{\theta\theta}'\chi}{\gamma_{rr}\gamma_{\theta\theta}} - \frac{2\alpha'\chi}{\gamma_{rr}r} + \frac{\alpha'\chi'}{2\gamma_{rr}} \\ & + \frac{\kappa_1(1-\kappa_2)\alpha\Theta}{\Omega} + \frac{2C_{Z4c}Z_r\chi\alpha'}{\gamma_{rr}} - \frac{\alpha\chi\gamma_{rr}'\Omega'}{2\gamma_{rr}^2\Omega} + \frac{\alpha\chi\gamma_{\theta\theta}'\Omega'}{\gamma_{rr}\gamma_{\theta\theta}\Omega} + \frac{3\chi\alpha'\Omega'}{\gamma_{rr}\Omega} - \frac{\alpha\chi'\Omega'}{2\gamma_{rr}\Omega} \\ & - \frac{2Z_r\alpha\chi\Omega'}{\gamma_{rr}\Omega} + \frac{\left[K+2(1-C_{Z4c})\Theta\right]\beta^r\Omega'}{\Omega} - \frac{2\alpha(K+2\Theta)}{a\Omega} + \frac{2\alpha\chi\Omega'}{\gamma_{rr}r\Omega} + \frac{\alpha\chi\Omega''}{\gamma_{rr}\Omega} \\ & + \frac{3\alpha}{a^2\Omega^2} - \frac{3\alpha\chi\left(\Omega'\right)^2}{\gamma_{rr}\Omega^2} + \frac{8\pi(\Pi-\beta^r\Phi')^2}{\alpha} \,,\end{aligned} \tag{20e}$$

Spherically symmetric equations

$$\dot{\chi} = \text{Principal part} + \ldots + \text{Z4 terms} + \frac{2\beta^r \chi \Omega'}{\Omega} - \frac{2\alpha\chi}{a\Omega} , \quad (20a)$$

$$\dot{\gamma_{rr}} = \text{Principal part} + \ldots , \quad (20b)$$

$$\dot{\gamma_{\theta\theta}} = \text{Principal part} + \ldots , \quad (20c)$$

$$\dot{A_{rr}} = \text{Principal part} + \ldots + \text{Z4 terms} + \frac{A_{rr}\beta^r \Omega'}{\Omega} - \frac{3\alpha A_{rr}}{a\Omega} + \Omega \text{ terms} + \text{Matter terms} , \quad (20d)$$

$$\dot{K} = \text{Principal part} + \ldots + \text{Z4 terms} + \frac{K\beta^r \Omega'}{\Omega} - \frac{2\alpha K}{a\Omega} + \Omega \text{ terms} + \text{Matter terms} , \quad (20e)$$

Spherically symmetric equations

$$\dot{\chi} = \text{Principal part} + \ldots + \text{Z4 terms} + \frac{2\beta^r\chi\Omega'}{\Omega} - \frac{2\alpha\chi}{a\Omega}\ , \tag{20a}$$

$$\dot{\gamma_{rr}} = \text{Principal part} + \ldots\ , \tag{20b}$$

$$\dot{\gamma_{\theta\theta}} = \text{Principal part} + \ldots\ , \tag{20c}$$

$$\dot{A_{rr}} = \text{Principal part} + \ldots + \text{Z4 terms} + \frac{A_{rr}\beta^r\Omega'}{\Omega} - \frac{3\alpha A_{rr}}{a\Omega} + \Omega\ \text{terms} + \text{Matter terms}\ , \tag{20d}$$

$$\dot{K} = \text{Principal part} + \ldots + \text{Z4 terms} + \frac{K\beta^r\Omega'}{\Omega} - \frac{2\alpha K}{a\Omega} + \Omega\ \text{terms} + \text{Matter terms}\ , \tag{20e}$$

$$\dot{\Lambda^r} = \text{Principal part} + \ldots + \text{Z4 terms} + \frac{\xi_{\Lambda^r}\Lambda^r\beta^{r\prime}}{a\Omega} + \Omega\ \text{terms} + \text{Matter terms}\ , \tag{20f}$$

$$\dot{\alpha} = \text{Principal part} + \ldots - \frac{3\alpha\beta^r\Omega'}{\Omega} + \Omega\ \text{terms}\ , \tag{20g}$$

$$\dot{\beta^r} = \text{Principal part} + \ldots - \frac{\xi_{\beta^r}\beta^r}{\Omega} + \Omega\ \text{terms}\ , \qquad \dot{B^r} = \text{Principal part} + \ldots\ , \tag{20h}$$

$$\dot{\Theta} = \text{Princ.} + \ldots + \frac{C_{Z4c}\Theta\beta^r\Omega'}{\Omega} + \frac{(3C_{Z4c} - 4)\alpha\Theta}{a\Omega} - \frac{\kappa_1(2+\kappa_2)\alpha\Theta}{\Omega} + \Omega\ \text{terms} + \text{Mat.}\ , \tag{20i}$$

$$\dot{\Phi} = \text{Principal part}\ , \tag{20j}$$

$$\dot{\Pi} = \text{Principal part} + \ldots + \text{Z4 terms} + \frac{3\Pi\beta^r\Omega'}{\Omega} - \frac{3\alpha\Pi}{a\Omega} + \Omega\ \text{terms}\ . \tag{20k}$$

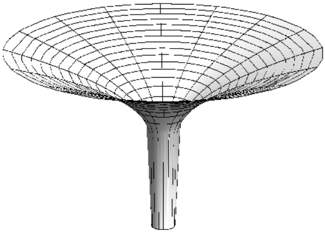
Schwarzschild spacetime on a hyperboloidal foliation

In strong field initial data, the parameter C_{CMC} determines the behaviour of the inner part of the slices.

- For large C_{CMC}, the polynomial

$$A(\tilde{r})+\left(\frac{K_{CMC}\,\tilde{r}}{3}+\frac{C_{CMC}}{\tilde{r}^2}\right)^2 \quad (21)$$

  has no roots;
- for small C_{CMC} the slice is not defined between the roots $R_1 < \tilde{r} < R_2$;
- the critical value of C_{CMC} will provide a trumpetgeometry, with a double root $\tilde{r} = R_0$.



Hannam et al, arXiv:0804.0628 [gr-qc]

The interior part of a trumpet slice asymptotes to an infinitely long cylinder, located at a finite value of the Schwarzschild radial coordinate (R_0).

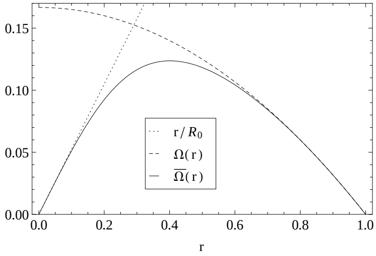
Schwarzschild hyperboloidal trumpet

For the critical C_{CMC} value, the compactification factor $\bar{\Omega}$ can be integrated numerically up to the origin of the new coordinate r by imposing conformal flatness and solving numerically the equation

$$\gamma_{rr0} = \frac{(\bar{\Omega} - r\,\bar{\Omega}')^2}{\left[A(\frac{r}{\bar{\Omega}}) + \left(\frac{K_{CMC}\,r}{3\bar{\Omega}} + \frac{C_{CMC}\bar{\Omega}^2}{r^2}\right)^2\right]\bar{\Omega}^2} = 1. \tag{22}$$

Two different asymptotic ends are compactified by $\bar{\Omega}$:

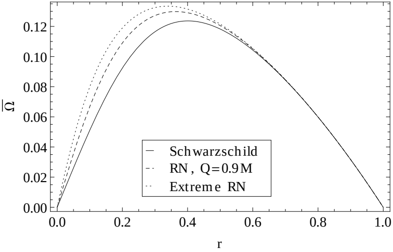
- Trumpet: $r = 0 \longleftrightarrow \tilde{r} = R_0$, where $\bar{\Omega} \sim r/R_0$.
- $\mathscr{I}^+$: $r = r_{\mathscr{I}} \longleftrightarrow \tilde{r} = \infty$, where $(\bar{\Omega} \sim \Omega)$.

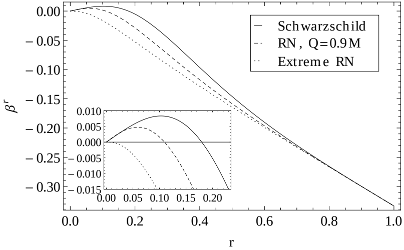


Reissner-Nordström

The properties of the Reissner-Nordström spacetime are interesting for what can be carried over to the Kerr spacetime case.

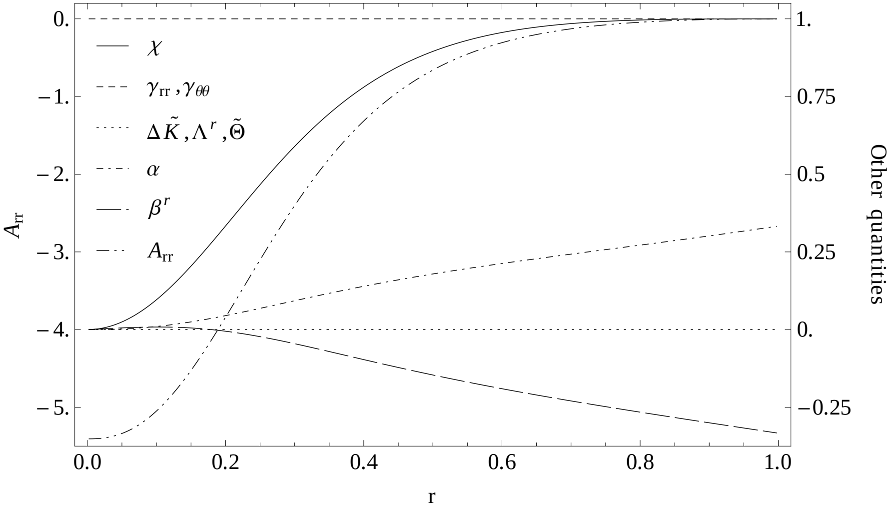
The Reissner-Nordström black hole slices present an equivalent dependence on the parameter C_{CMC} to the Schwarzschild case. Constant-mean-curvature Reissner-Nordström slices with critical C_{CMC} also correspond to trumpets.





The compactification factor $\bar{\Omega}$ varies in the interior part along with Q.

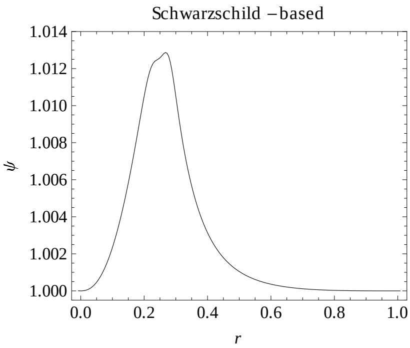
Schwarzschild trumpet initial data in the variables

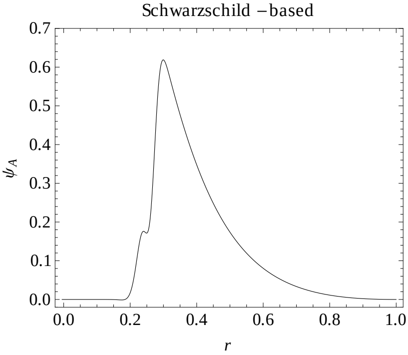


Solving the constraints

The Hamiltonian and momentum constraint equations have to be solved for the initial data including a scalar field perturbation.
The quantities used in the calculation are ψ and ψ_A, introduced as

$$\chi \to \chi_0 \psi^{-4} \qquad \text{and} \qquad A_{rr} \to (A_{rr0} + \psi_A)\psi^{-6}. \tag{23}$$





Preferred conformal gauge

The null tangent to $\mathscr{I}^+$ (l^a) is affinely parametrized, $l^a\nabla_a l^b|_{\mathscr{I}^+}=0$ (this does not imply an affine time coordinate), if

$$\Box\Omega\big|_{\mathscr{I}^+}=0. \tag{24}$$

This is the preferred conformal gauge, which also satisfies that the divergent terms in (2) independently attain regular limits at $\mathscr{I}^+$.

The preferred conformal gauge can be achieved by choosing equations of motion for the gauge variables that satisfy (24). A possibility is to choose $\dot\alpha$ and $\dot\beta^r$ calculated from

$$\tilde g^{ab}\left(\tilde\Gamma^c_{ab}-\hat{\tilde\Gamma}^c_{ab}\right)=\tilde F^c, \tag{25}$$

where $\tilde\Gamma^c_{ab}$ is the four-dimensional physical connection, $\hat{\tilde\Gamma}^c_{ab}$ is the connection calculated from a time-independent background metric and $\tilde F^c$ is a source function that has to satisfy $\tilde F^r|_{\mathscr{I}^+}\propto\Omega^q$ with $q>2$.

A posteriori reparametrization

An affinely parametrized time coordinate (Bondi time, t_B) is achieved if the preferred conformal gauge and $\alpha^2\big|_{\mathscr{I}^+} \propto \beta^r\big|_{\mathscr{I}^+}$ are satisfied. If this is not the case, the code time can be reparametrized a posteriori to obtain the Bondi time. Introducing a new conformal factor

$$\check{\Omega} = \omega\Omega, \tag{26}$$

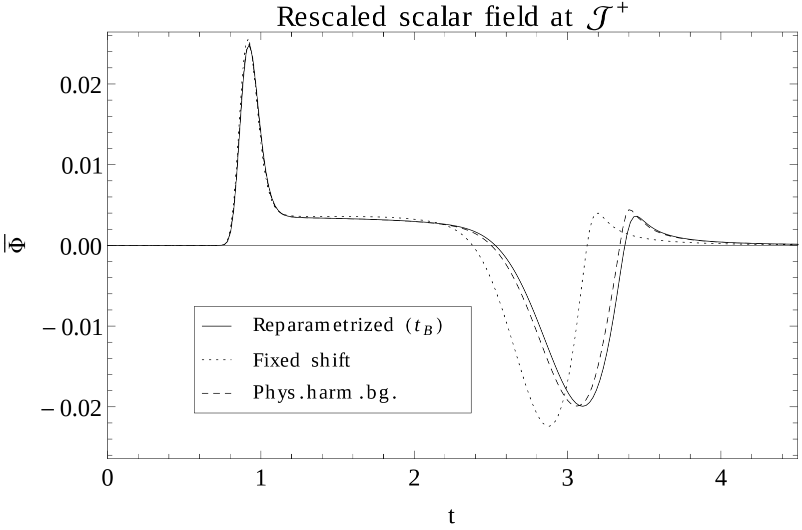
where $\check{\Omega}$ satisfies the preferred conformal gauge. The deviations from it are included in ω at $\mathscr{I}^+$ by the expression

$$\frac{\dot{\omega}}{\omega} = \frac{\beta^r \gamma_{rr}'}{2\gamma_{rr}} - \frac{\alpha\chi\alpha'}{\gamma_{rr}\beta^r} + \beta^{r\prime} - \frac{\beta^r\chi'}{2\chi}. \tag{27}$$

The relation between the Bondi and code times at $\mathscr{I}^+$ are given by

$$dt_B = \frac{\alpha^2\omega}{\beta^r\Omega'}dt. \tag{28}$$

Time reparametrization



Method of Lines

Discretizing all but the time coordinate transforms the original system of PDEs into a system of ODEs. Now any ODE integrator can be applied:

$$\vec{u}^{(j+1)} = \vec{u}^{(j)} + \Delta t \vec{f}(t_j, \vec{x}, \vec{u}^{(j)}, \partial_{\vec{x}}\vec{u}^{(j)}, \partial^2_{\vec{x}}\vec{u}^{(j)}, ...) \qquad \text{(Euler method).} \tag{29}$$

Explicit (CFL condition), implicit (expensive) or mixed algorithms.
Study stability of the discretization with a von Neumann analysis.

Finite Differences

Here preferred over pseudo-spectral methods (more difficult to stabilize).
Spatial derivatives are substituted by an approximation to their continuum value in terms of difference quotients:

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{u(x+\Delta x) - u(x-\Delta x)}{2\Delta x} \equiv \frac{u_{i+1} - u_{i-1}}{2\Delta x} + \mathcal{O}(\Delta x^2). \tag{30}$$

Stencils can be centered or asymmetric (one-sided, one point off-centered).

Simulation setup

The numerical code uses:

- Method of Lines
    - 4th order Runge-Kutta time integrator
    - Finite Differences of orders 2, 4, 6 or 8
- Kreiss-Oliger dissipation
- Staggered grid: $r = 0$ and $r = r_{\mathscr{I}}$ are avoided.
- Centered stencils at the boundaries with
    - $r = 0$: parity conditions (regular); extrapolation (trumpet).
    - $r = r_{\mathscr{I}}$: extrapolating conditions.

The implementation has been tested with regular initial data and Schwarzschild trumpet initial data, evolving:

- Gauge waves
- Scalar field perturbations

Scalar field on Schwarzschild spacetime - decay tails

A scalar field perturbation of a Schwarzschild black hole is expected to decay at late times with a fall-off form

$$\lim_{t\to+\infty} \Phi(t,r) \propto t^p \ . \tag{31}$$

In spherical symmetry the decay rate p takes the values:

- $p = -3$ along timelike surfaces and
- $p = -2$ along null surfaces ($\mathscr{I}^+$).

The drift experienced by the variables is very slow, so that the decay tails can be observed before the drift effects become relevant.