Universal metrics in modified theories of gravity

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Based on

S. Hervik, V. Pravda, A.P., *Type III and N universal spacetimes*, CQG **31**, 215005 (2014)

S. Hervik, T. Málek, V. Pravda, A.P., *Type II universal spacetimes*, CQG **32**, 245012 (2015)

Action $\rightarrow$ field equations $\rightarrow$ solutions

General Lagrangian theory

action $S \equiv \int \mathrm{d}^n x \sqrt{-g} \mathcal{L} \longrightarrow \delta S = 0 \longrightarrow$ field equations $\longrightarrow$ solutions

Einstein's gravity

$\int \mathrm{d}^n x \sqrt{-g} \frac{1}{\kappa} \left(R - 2\Lambda_0 \right) \longrightarrow R_{ab} - \frac{1}{2} R g_{ab} + \Lambda_0 g_{ab} = 0 \longrightarrow$ solutions

Quadratic gravity

$$S = \int \mathrm{d}^n x \sqrt{-g} \left(\frac{1}{\kappa} \left(R - 2\Lambda_0 \right) + \alpha R^2 + \beta R_{ab}^2 + \gamma \left(R_{abcd}^2 - 4R_{ab}^2 + R^2 \right) \right)$$

$\longrightarrow$ field equations [Gullu, Tekin, 2009]

$$\frac{1}{\kappa} \left(R_{ab} - \frac{1}{2} R g_{ab} + \Lambda_0 g_{ab} \right) + 2\alpha R \left(R_{ab} - \frac{1}{4} R g_{ab} \right) + (2\alpha + \beta) \left(g_{ab} \nabla^c \nabla_c - \nabla_a \nabla_b \right) R$$

$$+ 2\gamma \left(R R_{ab} - 2R_{acbd} R^{cd} + R_{acde} R_b{}^{cde} - 2R_{ac} R_b{}^c - \frac{1}{4} g_{ab} \left(R_{cdef}^2 - 4R_{cd}^2 + R^2 \right) \right)$$

$$+ \beta \nabla^c \nabla_c \left(R_{ab} - \frac{1}{2} R g_{ab} \right) + 2\beta \left(R_{acbd} - \frac{1}{4} g_{ab} R_{cd} \right) R^{cd} = 0.$$

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Universal spacetimes ⊂ CSI
Type N universal spacetimes
Type III universal spacetimes
Type II universal spacetimes

Universal spacetimes $\subset$ CSI

Lagrangian theories of gravity [Iyer, Wald, 1994]

action $S_G \equiv \int \mathrm{d}^n x \sqrt{-g} \mathcal{L}(g_{ab}, R_{abcd}, \nabla_{a_1} R_{bcde}, \ldots, \nabla_{a_1 \ldots a_k} R_{bcde}, \ldots)$
($\mathcal{L}$ = a polynomial curvature invariant) $\longrightarrow \delta S = 0 \longrightarrow$ conserved tensor

$$-T^{ab} = \frac{\partial L}{\partial g_{ab}} + E^a{}_{cde} R^{bcde} + 2\nabla_c \nabla_d E^{acdb} + \frac{1}{2} g^{ab} L$$

$$E^{bcde} = \frac{\partial L}{\partial R_{bcde}} - \nabla_{a_1} \frac{\partial L}{\partial \nabla_{a_1} R_{bcde}}$$

$$+ \cdots + (-1)^p \nabla_{(a_1} \cdots \nabla_{a_p)} \frac{\partial L}{\partial \nabla_{(a_1} \cdots \nabla_{a_p)} R_{bcde}}$$

$\longrightarrow$ vacuum field equations $\longrightarrow$ SOLUTIONS?
universal spacetimes are vacuum solutions to **all** such theories

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Universal spacetimes - definition

k-**universal** metric ($\in U_k$):

$\forall\ T_{ab} = T(g_{ab}, R_{abcd}, \nabla_{a_1}R_{bcde}, \ldots, \nabla_{a_1\ldots a_k}R_{bcde})$:

$T^{[ab]} = 0,\ T^{ab}{}_{;b} = 0\ \Rightarrow\ T_{ab} = \lambda g_{ab}$

universal metric ($\in U$): k-universal for all k

First examples of universal spacetimes:

- in the context of string theory

[Amati, Klimčík, 1989], [Horowitz, Steif, 1990] pp-waves

- as spacetimes with vanishing quantum corrections

[Coley, Gibbons, Hervik, Pope, 2008]

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Main results - Relation to CSI spacetimes

Proposition 1 - Universal spacetimes belong to CSI

$$\mathrm{U} \subset \mathrm{CSI}$$

(CSI: all curvature invariants
$I = I(g_{ab}, R_{abcd}, \ldots, \nabla^k R_{abcd}, \ldots) =$const.)

- conjectured in [Coley, Hervik, 2011]
- proven in [Hervik, Pravda, A.P., 2014]
- $\exists$ spacetimes $\in$ CSI and $\notin$ U

CSI condition is **not** sufficient for universality

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Type N universal spacetimes

Proposition 2a - type N universal spacetimes

A type N Einstein spacetime $\in U_0$.

Proposition 2b - type N universal spacetimes

A type N spacetime $\in U \iff$ it is an Einstein Kundt spacetime.

- proven in [Hervik, Pravda, A.P., 2014]
- sufficient part without a proof in [Coley, Gibbons, Hervik, Pope 2008]

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Type III universal spacetimes

For Kundt spacetimes, the WAND $\boldsymbol{\ell}$ is geodetic, expansion-free, shearfree and twist-free ($\rho_{ij} = 0$): $\ell_{a;b} = L_{11}\boldsymbol{\ell}_a\ell_b + \tau_i(\ell_a m_b^{(i)} + m_a^{(i)}\ell_b)$

- $\tau_i = 0$ recurrent ($\Rightarrow \lambda = 0$)

Proposition 3 - type III universal spacetimes

Type III Einstein Kundt spacetimes,
recurrent $\tau_i = 0$, $C_{acde}C_b{}^{cde} = 0$ $\implies$ U

- $\tau_i \neq 0$
  $\exists$ type III Einstein Kundt universal spacetimes with $\tau_i \neq 0$?

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5D

Type II - dimension-dependent!

$n = 5$:

- For universal spacetimes
$S_{ab}^{(2)} \equiv C_{acde} C_b{}^{cde}$
$S_{ab}^{(3)} \equiv C^{cdef} C_{cdga} C_{ef}{}^g{}_b$ conserved
- Necessary conditions for universality
$S_{ab}^{(2)} = K g_{ab}$
$S_{ab}^{(3)} = K' g_{ab}$
not compatible with 5D

Proposition 4 - 5D type II universal spacetimes

5D genuine type II 0-universal spacetimes $\nexists$

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Explicit examples for type D $n = Nn_0$, $n_0 \geq 2$

Proposition 5 - type D universal spacetimes $n = Nn_0$, $n_0 \geq 2$

$M = M_0 \times M_1 \times \cdots \times M_{N-1}$ (M_0 is Lorenzian)
M_α, $\alpha = 0 \ldots N-1$ are maximally symmetric spaces of dimension
n_α and the Ricci scalar R_α

- M is Einstein $\Longleftrightarrow \frac{R_\alpha}{n_\alpha} = \frac{R_0}{n_0}$, $\forall \alpha$
- M is universal $\Longleftrightarrow R_\alpha = R_0$, $n_\alpha = n_0$, $\forall \alpha$

- this leads to examples of universal spacetimes only for **composite**
number dimensions (Einstein spacetimes - any dim)

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Explicit examples for type II $n = Nn_0$, $n_0 \geq 4$

> ### Proposition 6 - type II universal spacetimes $n = Nn_0$, $n_0 \geq 4$
>
> When M_0 is Kundt proper Einstein ($\Lambda \neq 0$) genuine type N or III universal $\implies M$ - type II universal.

- this again leads to examples of universal spacetimes only for **composite** number dimensions
- no examples of type II universal spacetimes with **prime** numer dimensions are known
(recall that $\nexists$ type II universal spacetimes in 5D)

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A generalization of the Khlebnikov-Ghanam-Thompson metric (2-blocks) $n_0 = 2$

recurrent ($\ell_{a;b} = L_{11}\ell_a\ell_b$, $\ell = \mathrm{d}u$) type II Kundt metric

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v + (\lambda v^2 + H(u, x_\alpha, y_\alpha))\mathrm{d}u^2 + \frac{1}{|\lambda|}\sum_{\alpha=1}^{N-1}(\mathrm{d}x_\alpha^2 + s^2(x_\alpha)\,\mathrm{d}y_\alpha^2),$$

where $s(x_\alpha) = \sin(x_\alpha)$ for $\lambda > 0$,
$\qquad\quad s(x_\alpha) = \sinh(x_\alpha)$ for $\lambda < 0$
The metric is Einstein $\iff$

$$\Box H = \left[\sum_{\alpha=0}^{N-1}\Box^{(\alpha)}\right]H = 0, \quad \Box^{(\alpha)}H \equiv \nabla^{a_{(\alpha)}}\nabla_{a_{(\alpha)}}H$$

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Proposition 7 - on universality of 2-block KGT metric

Generalized 2-block KGT metric with $\left[\sum_{\alpha=0}^{N-1} \Box^{(\alpha)}\right] H = 0 \implies$ **0-universal**

If in addition $\left[\sum_{\alpha=0}^{N-1} (\Box^{(\alpha)})^2\right] H = 0 \implies$ **2-universal**.

- 0-universal $\implies$ vacuum solutions to all gr. theories with field equations that may contain $\nabla^k R_{ab}$ but do not contain $\nabla^k R_{abcd}$, e.g. Lovelock gravity (no $\nabla^k R_{ab}$, no $\nabla^k R_{abcd}$) or quadratic gravity (yes $\nabla^k R_{ab}$, no $\nabla^k R_{abcd}$)

- 2-universal - also solutions of vacuum equations of e.g. all $L(\text{Riemann})$ gravities

Conjecture 8 - universality of 2-block KGT metric

Generalized 2-block KGT metric with
$\left[\sum_{\alpha=0}^{N-1} (\Box^{(\alpha)})^P\right] H = 0, \ P = 1 \ldots N \implies$ **universal**.

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A generalization of the Khlebnikov-Ghanam-Thompson metric (3-blocks) $n_0 = 3$

not recurrent ($\ell_{a;b} = L_{11}\ell_a\ell_b + \tau_i(\ell_a m_b^{(i)} + m_a^{(i)}\ell_b)$, $\ell = \mathrm{d}u$) type II Kundt metric

$$
\begin{aligned}
\mathrm{d}s^2 &= 2\mathrm{d}u\mathrm{d}v + H(u, z, x_\alpha, y_\alpha, z_\alpha)\mathrm{d}u^2 + 2\frac{2v}{z}\mathrm{d}u\mathrm{d}z - \frac{2}{\lambda z^2}\mathrm{d}z^2 \\
&\quad -\frac{2}{\lambda}\sum_{\alpha=1}^{N-1}\left[\mathrm{d}x_\alpha^2 + sh_\alpha^2\left(\mathrm{d}y_\alpha^2 + s_\alpha^2\mathrm{d}z_\alpha^2\right)\right], \quad \lambda < 0
\end{aligned}
$$

where $s_\alpha = \sin(y_\alpha)$, $sh_\alpha = \sinh(x_\alpha)$
The metric is Einstein $\Longleftrightarrow$

$$\Box H - 2\lambda z H_{,z} = 0$$

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Proposition 9 - 0-universality of 3-KGT metric

Generalized 3-block KGT metric with $\left[\sum_{\alpha=1}^{N-1} \Box^{(\alpha)}\right] H = 0$,
$H = h_0(u, x_\alpha, y_\alpha, z_\alpha) + \frac{h_1(u, x_\alpha, y_\alpha, z_\alpha)}{z^2} \implies$ **0-universal**

Conjecture 10 - universality of 3-KGT metric

Generalized 3-block KGT metric with $H_{,z} = 0$, $\left[\sum_{\alpha=1}^{N-1} (\Box^{(\alpha)})^P\right] H = 0$,
$P = 1 \ldots N \implies$ **universal**

- Universal spacetimes are vacuum solutions of all theories with

$$\mathcal{L} = \mathcal{L}(g_{ab}, R_{abcd}, \nabla_{a_1} R_{bcde}, \ldots, \nabla_{a_1 \ldots a_p} R_{bcde})$$

- $U \subset$ CSI
- A type N spacetime $\in$ U $\Longleftrightarrow$ Einstein, Kundt
- A subclass of type III Einstein Kundt spacetimes $\subset$ U
- Explicit examples of type II universal metrics $\subset$ Kundt CSI class

$$\mathrm{d}s^2 = 2\mathrm{d}u\left[\mathrm{d}r + H(u, r, x^\gamma)\mathrm{d}u + W_\alpha(u, r, x^\gamma)\mathrm{d}x^\alpha\right] + g_{\alpha\beta}(x^\gamma)\mathrm{d}x^\alpha\mathrm{d}x^\beta,$$

where H - quadratic in r, W_α - linear in r and $g_{\alpha\beta}(x^\gamma)$ - a locally homogeneous metric.

- type D universal metrics $\exists$ for all **composite** number dimensions
- type N universal metrics can be used to construct type II universal metrics
- more general type II generalized KGT metrics may be also universal
- **no** examples of type II universal metrics for **prime** number dimensions are known
- 5D type II universal metrics **do not exist**
- n=7, 11, … ???

type	conditions	universality
N	Einstein	U_0
N	Einstein, Kundt	U
III	Einstein, Kundt, $\tau_i = 0$, $C_{acde}C_b{}^{cde} = 0$	U
III	Einstein, Kundt, $\tau_i \neq 0$	?U
II	$n = 5$	$\nexists$ U
II	$n = 7, 11, \ldots$	?U
D	direct of max. sym. $n = Nn_0$, $n_0 \geq 2$	U
II	direct of U(N,III) $\oplus$ max. sym., $n = Nn_0$, $n_0 \geq 4$	U
II	gen. KGT, $n = Nn_0$, $n_0 = 2$, additional conds.	U_2, ?U
II	gen. KGT, $n = Nn_0$, $n_0 = 3$, additional conds.	U_0, ?U
II	gen. KGT, $n = Nn_0$, additional conds.	?U